Solução de Sistemas Lineares: Métodos Diretos

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Decomposição LU

Teorema

Seja A uma matriz quadrada de ordem n, e A_k o menor principal k. Se $\det(A_k) \neq 0$, para $k = 1, 2, ..., n-1$. Então existe uma única matriz triangular inferior L e uma única matriz triangular superior U, tal que $A = LU$.

Teorema

Seja A uma matriz quadrada de ordem n, e A_k o menor principal k. Se $\det(A_k) \neq 0$, para $k = 1, 2, ..., n-1$. Então existe uma única matriz triangular inferior L e uma única matriz triangular superior U, tal que $A = LU$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{11} \end{pmatrix}$$

Decomposição LU

Teorema

Seja A uma matriz quadrada de ordem n, e A_k o menor principal k. Se $\det(A_k) \neq 0$, para $k = 1, 2, ..., n-1$. Então existe uma única matriz triangular inferior L e uma única matriz triangular superior U, tal que $A = LU$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{11} \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & & & \bigcirc \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & & u_{2n} \\ & & \ddots & \vdots \\ \bigcirc & & & u_{nn} \end{pmatrix}$$

Resolvendo um sistema linear

Resolver o sistema linear $Ax = b$, é equivalente a resolver os sistemas $Ly = b$ e $Ux = y$ consecutivamente, onde y é um vetor temporário.

$$\text{substituição progressiva} \quad \begin{pmatrix} 1 & & & \bigcirc \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\text{substituição regressiva} \quad \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & & u_{2n} \\ & & \ddots & \vdots \\ \bigcirc & & & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Resolvendo um sistema linear

- Pseudo-code
  - Substituição Progressiva

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} y_i}{l_{ii}}$$

  - Substituição Regressiva

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} u_{ij} x_i}{u_{ii}}$$

Substituição Progressiva

```
% Algoritmo de substituicoes progressivas para resolver Ly=b
% Input: Matriz quadrada triangular inferior L e vetor b.
% Output: Vetor solucao y.

function y = forwsub(L,b)
n = length(b);
y = zeros(n,1);
for i=1:n
    y(i) = (b(i)-L(i,1:i-1)*y(1:i-1,1))/L(i,i);
end
```

Substituição Regressiva

```
% Algoritmo de substituicoes regressivas para resolver Ux=y
% Input: Matriz quadrada triangular superior U e vetor y.
% Output: Vetor solucao x.

function x = backsub(U,y)
n = length(y);
x = zeros(n,1);
for i=n:-1:1
    x(i) = (y(i)-U(i,i+1:n)*x(i+1:n,1))/U(i,i);
end
```

Ordem de complexidade

- Substituição Progressiva: n^2 *flops*
- Substituição Regressiva: n^2 *flops*
- Decomposição LU: $\frac{2}{3}n^3$ *flops*
- Decomposição de Cholesky: $\frac{1}{3}n^3$ *flops*

- Pseudo-code

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}}{u_{jj}}$$

Decomposição LU

```
% Decomposicao LU sem pivoteamento
% Input: Matriz quadrada A(nxn).
% Output: Matrizes triangulares inferior L e superior U de A = LU.
function [L,U]=ludecomp(A)
n=size(A,1);
L=eye(n); U=zeros(n);
for k=1:n
    for j=k:n
        U(k,j)=A(k,j)-L(k,1:k-1)*U(1:k-1,j);
    end
    for i=k+1:n
        L(i,k)=(A(i,k)-L(i,1:k-1)*U(1:k-1,k))/U(k,k);
    end
end
```

```
% Resolver sistema Ax = b, usando decomposição LU
% Input: Matriz quadrada A e vetor b.
% Output: Vetor solução x.

function x = resolve_lu(A,b)
[L,U] = ludecomp(A);
y = forwsub(L,b);
x = backsub(U,y);
```

Exemplo

Seja:

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 1 & 3 \end{pmatrix}$$

1. Verificar se A satisfaz as condições de decomposição LU.
2. Decompor A em LU.
3. Calcular o determinante de A usando a decomposição obtida.
4. Resolver o sistema linear $Ax = b$, onde $b = (0, -7, -5)^t$.

Decomposição LU

1. Verificando que A satisfaz as condições de decomposição LU.
   $\det(A_1) = 5 \neq 0$ e $\det(A_2) = -1 \neq 0$
   Logo, A satisfaz as condições da decomposição LU.
2. $$LU = \begin{pmatrix} 1 & & \\ 3/5 & 1 & \\ 1/5 & -3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ & -1/5 & 17/5 \\ & & 13 \end{pmatrix}$$
3. $\det(A) = \det(LU) = 1 \cdot (\det U) = -13$
4. Para obter a solução do sistema linear $Ax = b$, devemos resolver dois sistemas lineares triangulares:
   $Ly = b$ e $Ux = y$.

Corolário

Se A é simétrica, positiva definda, então A pode ser decomposta unicamente no produto HH^t, onde H é matriz triangular inferior com elementos diagonais positivos.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

$$HH^t = \begin{pmatrix} h_{11} & & & & \bigcirc \\ h_{21} & h_{22} & & & \\ h_{31} & h_{32} & h_{33} & & \\ \cdots & \cdots & \cdots & \ddots & \\ h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn} \end{pmatrix} \begin{pmatrix} h_{11} & h_{21} & h_{31} & \cdots & h_{n1} \\ & h_{22} & h_{32} & \cdots & h_{n2} \\ & & h_{33} & \cdots & h_{n3} \\ & & & \ddots & \vdots \\ \bigcirc & & & & h_{nn} \end{pmatrix}$$

- Pseudo-code

$$h_{kk} = \sqrt{a_{kk} - \sum_{s=1}^{k-1} h_{ks}^2}$$

$$h_{kj} = \frac{a_{kj} - \sum_{s=1}^{j-1} h_{ks} h_{js}}{h_{jj}}$$

Decomposição de Cholesky

```
% Decomposicao de Cholesky
% Input: Matriz simetrica e positiva definida A(nxn)
% Output: Matriz de triangular inferior H

function H = choldecomp(A)
n = size(A,1);
H = tril(A);
for k=1:n-1
    H(k,k) = sqrt(H(k,k));
    H(k+1:n,k) = H(k+1:n,k)/H(k,k);
    for j=k+1:n
        H(j:n,j) = H(j:n,j)-H(j:n,k)*H(j,k);
    end
end
H(n,n) = sqrt(H(n,n));
```

```
% Resolver sistema Ax = b, usando Decomposição de Cholesky
% Input: Matriz quadrada A e vetor b.
% Output: Vetor solução x.

function x = resolve_chol(A,b)
H = choldecomp(A);
y = forwsub(H,b);
x = backsub(H',y);
```

Exemplo

Seja:

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 10 & 4 \\ -4 & 4 & 9 \end{pmatrix}$$

1. Verificar se A satisfaz as condições do método de Cholesky.
2. Decompor A em HH^t.
3. Calcular o determinante de A usando a decomposição obtida.
4. Resolver o sistema linear $Ax = b$, onde $b = (0, 6, 5)^t$.

Decomposição de Cholesky

1. A matriz A é simétrica. Verificando se é positiva definida.
   $\det(A_1) = 4 > 0, \quad \det(A_2) = 36 > 0, \quad \det(A_3) = 36 > 0.$
   Logo, A satisfaz as condições da decomposição HH^t.

2. $$H = \begin{pmatrix} 2 & & \bigcirc \\ 1 & 3 & \\ -2 & 2 & 1 \end{pmatrix}$$

3. $\det(A) = \det(HH^t) = (\det H)^2 = (h_{11}h_{22}h_{33})^2 = (2 \times 3 \times 1)^3 = 36.$

4. Para obter a solução do sistema linear $Ax = b$, devemos resolver dois sistemas lineares triangulares:
   $Hy = b$ e $H^t x = y$.

- Passar de $Ax = b$ para $\tilde{A}x = \tilde{b}$ usando o algoritmo:

$$A_i^{(k+1)} \leftarrow A_i^{(k)} - m_{ik} A_k^{(k)}, \text{ com } m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$$

$$k = 1, ..., n-1$$

$$i = k+1, ..., n$$

onde $\tilde{A}$ é uma matriz triangular superior.

- Resolver $\tilde{A}x = \tilde{b}$ usando o algoritmo de substituição regressiva:

$$x_k = \left(\tilde{b}_k - \sum_{j=K+1}^{n} \tilde{a}_{kj} x_j \right) / \tilde{a}_{kk}, \; k = n, ..., 1$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad A_2^{(2)} \leftarrow A_2^{(1)} - m \cdot A_1^{(1)}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad \begin{array}{l} A_2^{(2)} \leftarrow A_2^{(1)} - m \cdot A_1^{(1)} \\ b_2^{(2)} \leftarrow b_2^{(1)} - m * b_1^{(1)} \end{array}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\
a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\
a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)}
\end{array}\right]
$$

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$$m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\
a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\
a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)}
\end{array}\right]
$$

$$
m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}} \quad A_3^{(2)} \leftarrow A_3^{(1)} - m \cdot A_1^{(1)}
$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \color{red}{b_1^{(1)}} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & \color{blue}{b_3^{(1)}} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}} \quad \begin{array}{l} A_3^{(2)} \leftarrow A_3^{(1)} - m \cdot A_1^{(1)} \\ b_3^{(2)} \leftarrow \color{blue}{b_3^{(1)}} - m \cdot \color{red}{b_1^{(1)}} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}} \quad A_4^{(2)} \leftarrow A_4^{(1)} - m \cdot A_1^{(1)}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \color{red}{b_1^{(1)}} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & \color{blue}{b_4^{(1)}} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}} \quad \begin{array}{l} A_4^{(2)} \leftarrow A_4^{(1)} - m \cdot A_1^{(1)} \\ b_4^{(2)} \leftarrow \color{blue}{b_4^{(1)}} - m \cdot \color{red}{b_1^{(1)}} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right]$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_{1}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_{2}^{(2)} \\
0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_{3}^{(2)} \\
0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_{4}^{(2)}
\end{array}\right]
$$

$$
m = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} \quad \begin{array}{l} A_3^{(3)} \leftarrow A_3^{(2)} - m \cdot A_2^{(2)} \\ b_3^{(3)} \leftarrow b_3^{(2)} - m \cdot b_2^{(2)} \end{array}
$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right]$$

$$m = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} \quad \begin{array}{l} A_4^{(3)} \leftarrow A_4^{(2)} - m \cdot A_2^{(2)} \\ b_4^{(3)} \leftarrow b_4^{(2)} - m \cdot b_2^{(2)} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & a_{43}^{(3)} & a_{44}^{(3)} & b_4^{(3)} \end{array}\right]$$

$$m = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} \quad \begin{array}{l} A_4^{(4)} \leftarrow A_4^{(3)} - m \cdot A_3^{(3)} \\ b_4^{(4)} \leftarrow b_4^{(3)} - m \cdot b_3^{(3)} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & 0 & a_{44}^{(4)} & b_4^{(4)} \end{array}\right]$$

```
% Resolver sistema Ax = b, usando Eliminacao de Gauss
% Input: Matriz quadrada A e vetor b.
% Output: Vetor solução x.
function x=eliminacao_gauss(A,b)
n=size(A,1);
for k=1:n-1
    for i=k+1:n
        m = A(i,k)/A(k,k);
        A(i,:) = A(i,:) - m*A(k,:);
        b(i) = b(i) - m*b(k);
    end
end
x = backsub(A,b);
```

Exemplo1:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

$$\left\{\begin{array}{rcrcrcr} 6x_1 & + & 2x_2 & - & x_3 & = & 7 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 7 \\ 3x_1 & + & 2x_2 & + & 8x_3 & = & 13 \end{array}\right. \quad (1)$$

Cuja solução é

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2)$$

Exemplo1:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

$$\left\{ \begin{array}{rcrcrcr} 6x_1 & + & 2x_2 & - & x_3 & = & 7 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 7 \\ 3x_1 & + & 2x_2 & + & 8x_3 & = & 13 \end{array} \right. \quad (1)$$

Cuja solução é

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2)$$

Exemplo2:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

$$A = \begin{pmatrix} 10^{-18} & 1 \\ 1 & 1 \end{pmatrix} \tag{3}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4}$$

Cuja solução é

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5}$$

Exemplo2:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

$$A = \begin{pmatrix} 10^{-18} & 1 \\ 1 & 1 \end{pmatrix} \tag{3}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4}$$

Cuja solução é

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5}$$

- Pseudo-code

Encontrar pivô

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

Trocar linhas

$$L_k \longleftrightarrow L_p$$

Fazer Eliminação de Gauss

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad A_2^{(2)} \leftarrow A_2^{(1)} - m \cdot A_1^{(1)}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad \begin{array}{l} A_2^{(2)} \leftarrow A_2^{(1)} - m \cdot A_1^{(1)} \\ b_2^{(2)} \leftarrow b_2^{(1)} - m * b_1^{(1)} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & b_3^{(1)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}} \quad A_3^{(2)} \leftarrow A_3^{(1)} - m \cdot A_1^{(1)}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \color{red}{b_1^{(1)}} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\
a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} & \color{blue}{b_3^{(1)}} \\
a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)}
\end{array}\right]
$$

$$
m = \frac{a_{31}^{(1)}}{a_{11}^{(1)}} \quad
\begin{array}{l}
A_3^{(2)} \leftarrow A_3^{(1)} - m \cdot A_1^{(1)} \\
b_3^{(2)} \leftarrow \color{blue}{b_3^{(1)}} - m \cdot \color{red}{b_1^{(1)}}
\end{array}
$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}} \quad A_4^{(2)} \leftarrow A_4^{(1)} - m \cdot A_1^{(1)}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ a_{41}^{(1)} & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} & b_4^{(1)} \end{array}\right]$$

$$m = \frac{a_{41}^{(1)}}{a_{11}^{(1)}} \quad \begin{array}{l} A_4^{(2)} \leftarrow A_4^{(1)} - m \cdot A_1^{(1)} \\ b_4^{(2)} \leftarrow b_4^{(1)} - m \cdot b_1^{(1)} \end{array}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_{1}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_{2}^{(2)} \\
0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_{3}^{(2)} \\
0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_{4}^{(2)}
\end{array}\right]
$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & \color{red}{a_{22}^{(2)}} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ 0 & \color{blue}{a_{42}^{(2)}} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right]$$

Suponha $\color{red}{|a_{22}^{(2)}|} \ll |a_{32}^{(2)}| < \color{blue}{|a_{42}^{(2)}|}$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right]$$

$$A_{temp} \leftarrow A_4^{(2)} \qquad b_{temp} \leftarrow b_4^{(2)}$$

$$A_4^{(2)} \leftarrow A_2^{(2)} \qquad b_4^{(2)} \leftarrow b_2^{(2)}$$

$$A_2^{(2)} \leftarrow A_{temp} \qquad b_2^{(2)} \leftarrow b_{temp}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\
0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\
0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)}
\end{array}\right]
$$

$$
m = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} \quad
\begin{array}{l}
A_3^{(3)} \leftarrow A_3^{(2)} - m \cdot A_2^{(2)} \\
b_3^{(3)} \leftarrow b_3^{(2)} - m \cdot b_2^{(2)}
\end{array}
$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_{1}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_{2}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_{3}^{(3)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_{4}^{(2)} \end{array}\right]$$

$$m = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} \quad \begin{array}{l} A_4^{(3)} \leftarrow A_4^{(2)} - m \cdot A_2^{(2)} \\ b_4^{(3)} \leftarrow b_4^{(2)} - m \cdot b_2^{(2)} \end{array}$$

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & a_{43}^{(3)} & a_{44}^{(3)} & b_4^{(3)} \end{array}\right]$$

$$m = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} \quad \begin{array}{l} A_4^{(4)} \leftarrow A_4^{(3)} - m \cdot A_3^{(3)} \\ b_4^{(4)} \leftarrow b_4^{(3)} - m \cdot b_3^{(3)} \end{array}$$

$$
\left[\begin{array}{cccc:c}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & b_1^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\
0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\
0 & 0 & 0 & a_{44}^{(4)} & b_4^{(4)}
\end{array}\right]
$$

Eliminação de Gauss com pivoteamento

```
function x = eliminacao_gauss_pivo(A,b)
[m,n] = size(A);
Ab = [A b];

for k=1:n-1
    %parte do pivoteamento
    [~,p] = max(abs(Ab(k:n,k)));
    p = p+k-1;
    if p ~= k
        % pivoteamento das linhas
        Ab([k,p],:) = Ab([p,k],:)
    end
    for i=k+1:n
        m = Ab(i,k)/Ab(k,k);
        Ab(i,k:end) = Ab(i,k:end)-m*Ab(k,k:end);
    end
end

x = backsub(Ab(:,1:n),Ab(:,end));
```

Exemplo2:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

$$A = \begin{pmatrix} 10^{-18} & 1 \\ 1 & 1 \end{pmatrix} \tag{6}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{7}$$

Cuja solução é

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

Exemplo2:

Resolver, usando o método de Eliminação de Gauss, o sistema linear:

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