Introduction to Computational Fluid Dynamics

Gustavo C. Buscaglia

Graduate course

ICMC-USP, São Carlos, Brasil
gustavo.buscaglia@gmail.com

1 Principles and equations of Fluid Mechanics

1.1 Continuous media

- The continuum hypothesis.
- What is a material point?
- The velocity.

1.2 Cartesian vectors and tensors

We assume $\{x_1, x_2, x_3\}$ to be Cartesian coordinates, with

$$\check{e}^{(1)}, \quad \check{e}^{(2)}, \quad \check{e}^{(3)} \tag{1.1}$$

the Cartesian basis of vectors.

Vector field:

$$\mathbf{u}(\mathbf{x}, t) = \sum_i u_i(\mathbf{x}, t)\, \check{e}^{(i)} \tag{1.2}$$

Gradient:

$$\nabla\varphi = \sum_i \frac{\partial \varphi}{\partial x_i} \check{e}^{(i)} = \varphi_{,i}\, \check{e}^{(i)} \tag{1.3}$$

$$\underline{\nabla\varphi} = (\varphi_{,1}, \varphi_{,2}, \varphi_{,3})^T \tag{1.4}$$

Divergence:

$$\nabla \cdot \mathbf{u} = \sum_i \frac{\partial u_i}{\partial x_i} = u_{i,i} \tag{1.5}$$

Tensor product of two vectors:

$$\mathbf{u} \otimes \mathbf{v} = \sum_{i,j} u_i v_j \check{e}^{(i)} \otimes \check{e}^{(j)} \tag{1.6}$$

$$(\mathbf{u} \otimes \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}\,(\mathbf{v} \cdot \mathbf{w}) \tag{1.7}$$

Double contraction:

$$(\mathbf{u} \otimes \mathbf{v}) : (\mathbf{w} \otimes \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) = \sum_{i,j} u_i v_j w_i z_j \tag{1.8}$$

$$\mathbf{T} : \mathbf{S} = \sum_{i,j} T_{ij} S_{ij} \tag{1.9}$$

Gradient of a vector field:

$$\nabla \mathbf{u} = \sum_{i,j} u_{i,j} \check{e}^{(i)} \otimes \check{e}^{(j)} \tag{1.10}$$

$$\left(\underline{\underline{\nabla \mathbf{u}}}\right)_{ij} = u_{i,j} \tag{1.11}$$

Theorem 1.1 *Volume integral of a gradient.*

$$\int_V \varphi_{,i} \, dV = \int_{\partial V} \varphi \, n_i \, dS \tag{1.12}$$

Theorem 1.2 *Gauss-Green,* $\check{\mathbf{n}}$ *is the outward normal.*

$$\int_V \nabla \cdot \mathbf{z} \, dV = \int_{\partial V} \mathbf{z} \cdot \check{\mathbf{n}} \, dS \tag{1.13}$$

Outer product, cross product:

$$\mathbf{w} \times \mathbf{z} = \varepsilon_{ijk}\, w_j\, z_k\, \check{\mathbf{e}}^{(i)} \tag{1.14}$$

Curl of a vector:

$$\nabla \times \mathbf{z} = \varepsilon_{ijk}\, z_{k,j}\, \check{\mathbf{e}}^{(i)} \tag{1.15}$$

Exo. 1.1 *Show that the divergence of $\nabla \times \mathbf{z}$ is zero, for any differentiable vector field $\mathbf{z}$. Show that the curl of $\nabla\varphi$ is zero, for any differentiable scalar function φ.*

Exo. 1.2 *Let V be a connected volume in 3D, with boundary ∂V. Assume that the fluid inside V is at constant pressure, exerting a force*

$$\mathbf{F} = p\, \check{\mathbf{n}} \tag{1.16}$$

per unit area on ∂V. Prove that the total force exerted by the inner fluid on the boundary is zero.

Exo. 1.3 *Let V be a volume in 3D, with boundary ∂V. Assume the volume is filled with a fluid of constant density ρ. Prove that the total weight can be obtained from surface integrals:*

$$\int_V \rho\, g\; dV = \frac{\rho\, g}{3} \int_{\partial V} \mathbf{x} \cdot \check{\mathbf{n}}\; dS = \rho\, g \int_{\partial V} x_3\, n_3\; dS \tag{1.17}$$

Exo. 1.4 *Prove Archimedes' principle. A body immersed in a stagnant homogeneous liquid (which has pressure proportional to its depth, $p = \rho\, g\, h$) experiences a net upward force equal to the weight of the displaced liquid.*

1.3 Kinematics, material derivative and transport theorem

The trajectory of particles in a continuum can be described by a function $\mathcal{F}(\mathbf{x}, s, t)$ which gives *the position at time t of the particle that ocupies position $\mathbf{x}$ at time s.*

- $\mathcal{F}(\mathbf{x}, t, t) = \mathbf{x}$ for all t.
- Fixing s and t, considered just as function of $\mathbf{x}$, the function $\boldsymbol{\phi}(\mathbf{x}) = \mathcal{F}(\mathbf{x}, s, t)$ is the *deformation* field of the medium between times s and t.
- The velocity field is related to $\mathcal{F}$

$$\frac{\partial \mathcal{F}}{\partial t}(\mathbf{x}, s, t) = \mathbf{u}(\mathcal{F}(\mathbf{x}, s, t), t) \tag{1.18}$$

  Here the pair $(\mathbf{x}, s)$ are a label for the *particle*. Another usual label is $\mathbf{X}$, defined as the position occupied by the particle in some "reference configuration", which needs not correspond to an instant of time. This is the so-called Lagrangian frame.

- Trajectories are sometimes written as

$$\mathbf{x}(t) = \boldsymbol{\phi}(\mathbf{X}, t) \tag{1.19}$$

- Pathlines, streamlines and streaklines.

Exo. 1.5 *A continuum is rigidly rotating with angular velocity ω around the axis $\mathbf{a} = \check{\mathbf{e}}^{(1)} + \check{\mathbf{e}}^{(2)}$. Compute its Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ and its kinematic history function $\mathcal{F}(\mathbf{x}, s, t)$.*

The *material* or *total* derivative of a quantity ψ at time t for the particle that at that time is located at $\mathbf{x}$ is defined as the "derivative following the particle", or, more precisely,

$$\frac{D\psi}{Dt} = \lim_{\delta \to 0} \frac{\psi(\mathcal{F}(\mathbf{x}, t, t+\delta), t+\delta) - \psi(\mathbf{x}, t)}{\delta} \tag{1.20}$$

Exo. 1.6 *Prove that*

$$\frac{D\psi}{Dt} = \partial_t \psi + \mathbf{u} \cdot \nabla \psi \tag{1.21}$$

The *acceleration* of a fluid is the material derivative of the velocity

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \partial_t \mathbf{u} + (\nabla \mathbf{u}) \cdot \mathbf{u} \tag{1.22}$$

Exo. 1.7 *Compute the acceleration field of the rigid rotation described in Exo. 1.5.*

Let Ω be a region in space, and let $f(\mathbf{x}, t)$ be a scalar field defined in Ω. To fix ideas, let f be a *temperature* field.

Let us select, at time t, a region V of Ω. This defines a *material volume*, consisting of the set of material particles that are inside V at time t.

If one follows the particles that are in V at t, they will occupy another region of space $\mathcal{V}(t')$ at time t'. Obviously $\mathcal{V}(t) = V$.

For any t', let $I(t')$ be the integral of f, at time t', over the volume occupied $\mathcal{V}(t')$ by the particles

$$I(t') = \int_{\mathcal{V}(t')} f(\mathbf{x}, t') \; dV \; . \tag{1.23}$$

Clearly $I(t')$ is the integral of the temperature over the material volume, a volume that changes position in time but has fixed material identity.

Reynolds transport theorem.

$$\frac{DI}{Dt}(t) = \int_V \left[\partial_t f + \nabla \cdot (\mathbf{u}\, f)\right] \; dV = \int_V \partial_t f \; dV + \int_{\partial V} f \, \mathbf{u} \cdot \check{\mathbf{n}} \; dS \tag{1.24}$$

Exo. 1.8 *Use the previous formula to prove that a flow in which the volume of each material part is preserved must be solenoidal ($\nabla \cdot \mathbf{u} = 0$), also called incompressible.*

Computational exercise:

- Consider a structured mesh in space-time: $\{x_i\} \times \{y_j\} \times \{t_k\}$. Consider that a velocity vector is known on each node and time of the mesh: $\{\mathbf{u}_{ij}^k\}$.
- A velocity field $\mathbf{u}(\mathbf{x}, t)$ is defined by **trilinear interpolation** of the instantaneous nodal velocity vectors.
- Consider also that a set of points $\{\mathbf{X}_m\}$ is given.

Build an Octave code that calculates the trajectories of particles that, at time t_0, are in the positions $\{\mathbf{X}_m\}$. Plot and animate in an interesting example.

1.4 Conservation of mass

Let M be the mass contained at time t in volume V,

$$M = \int_V \rho \, dV \ . \tag{1.25}$$

Since *the mass is conserved*,

$$\frac{DM}{Dt} = 0 \ , \tag{1.26}$$

which implies that **(integral form)**

$$\int_V \partial_t \rho \, dV = - \int_{\partial V} \rho \, \mathbf{u} \cdot \check{\mathbf{n}} \, dS \tag{1.27}$$

and also that **(differential form)**

$$\partial_t \rho + \nabla \cdot (\rho \, \mathbf{u}) = 0 \tag{1.28}$$

This last equation can be written as

$$\frac{D\rho}{Dt} + \rho \, \nabla \cdot \mathbf{u} = 0 \ , \tag{1.29}$$

which shows that an incompressible flow ($\nabla \cdot \mathbf{u} = 0$) in which the density of the material particles does not change with time automatically satisfies mass conservation.

The **mass flux** is given by

$$\mathbf{j} = \rho\,\mathbf{u}\ . \tag{1.30}$$

The conservation of mass can be written as a *conservation law*:

$$\partial_t \rho + \nabla \cdot \mathbf{j} = g \tag{1.31}$$

where g represents the sources (in the case of mass equal to zero).

$$\frac{d}{dt}\int_V \rho\ dV = -\int_{\partial V} \underbrace{\mathbf{j}\cdot\check{\mathbf{n}}}_{J}\ dS\ +\ \int_V g\ dV \qquad \text{variation = inflow - outflow + internal sources} \tag{1.32}$$

Exo. 1.9 *Let ψ be the* mass density*, or* mass fraction*, of some species A dispersed in the medium. The mass of this species in some volume V is*

$$M_A = \int_V \rho\,\psi\ dV\ . \tag{1.33}$$

Derive conservation laws in differential and integral form for ψ. Also prove that

$$\frac{D\psi}{Dt} = 0\ . \tag{1.34}$$

1.5 Conservation of momentum

The total momentum contained by a region V of a continuum is

$$\mathbf{P} = \int_V \rho\,\mathbf{u}\; dV \;. \tag{1.35}$$

The principle of conservation of momentum states that changes in the momentum are equal to the applied (volumetric and surface) forces, i.e.

$$\frac{D\mathbf{P}}{Dt} = \int_V \mathbf{f}\; dV \;+\; \int_S \mathbf{F}\; dS \;. \tag{1.36}$$

Using the transport theorem one arrives at the integral form

$$\boxed{\frac{d}{dt}\int_V \rho\,\mathbf{u}\; dV \;= \int_V \mathbf{f}\; dV \;+\; \int_{\partial V} \left[\mathbf{F} - \rho\,(\mathbf{u}\otimes\mathbf{u})\,\check{\mathbf{n}}\right]\; dS \;.} \tag{1.37}$$

The Cauchy stress tensor

The *action-reaction principle* requires that, if at a point $\mathbf{x}$ of ∂V the region is subject to a surface force density $\mathbf{F}(\mathbf{x})$, the continuum inside reacts with an equal and opposite force.
It can be proved that there exists a symmetric tensor, the *Cauchy stress tensor*, such that for all $\mathbf{x}$ and t

$$\mathbf{F}(\mathbf{x},t) = \boldsymbol{\sigma}(\mathbf{x},t)\cdot\check{\mathbf{n}}(\mathbf{x},t)\ , \tag{1.38}$$

in the sense that *the surface forces that a medium exerts on another body through a surface with normal* $\mathbf{n}$ *(pointing outwards) is equal to* $-\boldsymbol{\sigma}\cdot\check{\mathbf{n}}$.

Inserting the stress tensor in (1.37) one arrives at

$$\frac{d}{dt}\int_V \rho\,\mathbf{u}\,dV \;=\int_V \mathbf{f}\,dV \;+\; \int_{\partial V}(\boldsymbol{\sigma}-\rho\,\mathbf{u}\otimes\mathbf{u})\cdot\check{\mathbf{n}}\,dS\ . \tag{1.39}$$

The momentum flux through a surface is, thus,

$$\boldsymbol{\zeta} = -\boldsymbol{\sigma} + \rho\,\mathbf{u}\otimes\mathbf{u} \tag{1.40}$$

Exo. 1.10 *From (1.39) deduce the following differential forms of momentum conservation:*

Conservative form:

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot\boldsymbol{\zeta} = \mathbf{f} \qquad \textit{or} \tag{1.41}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} \tag{1.42}$$

Non-conservative form:

$$\rho\,\partial_t\mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} \tag{1.43}$$

Also, write the equations above in Cartesian components.

1.6 Conservation of energy

Exo. 1.11 *Read 1.6 and 1.7 from Wesseling.*

The energy of a part of a continuum which occupies volume V is

$$E = \int_V \rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \, dV \tag{1.44}$$

where e is the *internal energy per unit mass*, which expresses the capability of a medium storing energy and is a function of its *local state*. The principle of conservation of energy reads

$$\frac{DE}{Dt} = \mathcal{Q} + \mathcal{W} \, , \tag{1.45}$$

where the right-hand side is the sum of the heat and work received from the surroundings.
Defining $\mathbf{q}$ as the heat flux and Q as the heat source per unit volume one gets

$$\frac{DE}{Dt} = \int_V (\mathbf{f} \cdot \mathbf{u} + Q) \, \, dV + \int_{\partial V} (\mathbf{u} \cdot \boldsymbol{\sigma} - \mathbf{q}) \cdot \check{\mathbf{n}} \, dS \tag{1.46}$$

Exo. 1.12 *From the equation above, prove the following differential form*

$$\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} + \boldsymbol{\sigma} : \nabla \mathbf{u} + Q \tag{1.47}$$

1.7 Constitutive laws

If one counts the equations up to now we have

- Conservation of mass (1 equation).
- Conservation of momentum (3 equations).
- Conservation of energy (1 equation).

Total: **5 equations**.

Counting the unknowns: ρ (1), $\mathbf{u}$ (3), $\boldsymbol{\sigma}$ (6), e (1), $\mathbf{q}$ (3). Total: **14 unknowns**.

The 9 equations that are lacking come from the so-called *constitutive laws*, that describe the material behavior (notice that the equations up to now hold for *any* continuum).

Essentially we need laws for e, $\boldsymbol{\sigma}$ and $\mathbf{q}$. For the latter Fourier's law is almost universally adopted,

$$\mathbf{q} = -\,\boldsymbol{\kappa}\,\nabla T\ , \tag{1.48}$$

where T is the temperature and $\boldsymbol{\kappa}$ the thermal conductivity (in general a tensor).

1.8 Newtonian and quasi-newtonian behavior

- The stress of a fluid at a point $\mathbf{x}$ and instant t can in principle depend on the whole deformation history of the vicinity of $\mathbf{x}$.

- However, not all constitutive laws correspond to fluids. The definition of fluid requires that "if the vicinity of the point has not deformed at all, then the stress tensor must be spherical". Spherical, in this context, means that $\boldsymbol{\sigma}$ is a multiple of the identity.

- A most important class of fluid constitutive laws corresponds to the so-called *quasi-Newtonian fluids*:

$$\boldsymbol{\sigma} = (-p + \lambda \nabla \cdot \mathbf{u})\,\mathbf{1} + \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \tag{1.49}$$

  in which λ and μ can depend on the *instantaneous deformation rate tensor*

$$\boldsymbol{\varepsilon}(\mathbf{u}) = D\,\mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \ . \tag{1.50}$$

- Since λ and μ are scalars, the model is *objective* only if they depend on $\boldsymbol{\varepsilon}(\mathbf{u})$ through is *invariants*:

$$I = \operatorname{trace} \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{1} : \boldsymbol{\varepsilon}(\mathbf{u}) = \nabla \cdot \mathbf{u} \tag{1.51}$$

$$II = \frac{1}{2}\left[(\operatorname{trace} \boldsymbol{\varepsilon}(\mathbf{u}))^2 - \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})\right] \tag{1.52}$$

$$III = \det \boldsymbol{\varepsilon}(\mathbf{u}) \tag{1.53}$$

  Notice that, in particular, the *deformation rate*

$$\|\boldsymbol{\varepsilon}(\mathbf{u})\| = \sqrt{\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})} \tag{1.54}$$

- If λ and μ are constants, eventually dependent on the temperature, the fluid is called *Newtonian*.

- Shear thinning (resp. shear thickening) describe fluids in which μ is a decreasing (resp. increasing) function of $\|\boldsymbol{\varepsilon}(\mathbf{u})\|$.

Exo. 1.13 *Knowing that the velocity field of a rigid body motion is given by*

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{z}(t) + \mathbf{r}(t) \times \mathbf{x} \ , \tag{1.55}$$

1. *Prove that* $\boldsymbol{\varepsilon}(\mathbf{u})$ *is zero.*

2. *Compute the* **vorticity** $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ *and find its relation to* $\mathbf{r}$ *and to the* **antisymmetric part of the velocity gradient**, $\nabla^A \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right)$.

Exo. 1.14 *For an incompressible fluid, the term* $\Phi = \boldsymbol{\sigma} : \nabla \mathbf{u}$ *in the differential equation* **dissipation** *of energy, i.e., the power transformed into heat. Write down* Φ *in Cartesian coordinates.*

1.9 Boundary conditions

Exo. 1.15 *Read 1.6 from Kirby.*

Exo. 1.16 *Read, fill in the details and reproduce (part of) the results of the articles by N. Morhell and H. Pastoriza (*Microfluidics and Nanofluidics*, 2013,* Sensors and Actuators B*, 2016).*

2 Brief overview of numerical methods for CFD

For this chapter we follow basically the two references:

- Finite Volume Methods. R. Eymard, T. Gallouët and R. Herbin. 2003. Pages 4-26, and also some small parts of Chapter 3.
- Principles of Computational Fluid Dynamics. P. Wesseling. 2001. Chapter 3.

2.1 Differential, integral and variational formulations

Consider the general second-order differential equation

$$L\varphi = -(\mathrm{a}_{ij}\varphi_{,j})_{,i} + (\mathrm{b}_i\varphi)_{,i} + c\,\varphi = q \ . \tag{2.1}$$

This equation is said to be *uniformly elliptic* if there exists $C > 0$ such that

$$\mathbf{v}\cdot(\mathbf{a}(\mathbf{x})\cdot\mathbf{v}) = \mathrm{a}_{ij}(\mathbf{x})\,\mathrm{v}_i\mathrm{v}_j \geq C\,\|\mathbf{v}\|^2 \qquad\qquad \forall\,\mathbf{x}\ \ \forall\,\mathbf{v}\ . \tag{2.2}$$

This condition, together with suitable boundary conditions, guarantees the existence of a unique φ in the space $H^1(\Omega)$. This solution is continuous (a.e.) across any surface.

Equation (2.1) can be seen as a steady conservation law in differential formulation,

$$\nabla\cdot\mathbf{j} = g \ , \tag{2.3}$$

by taking

$$\mathbf{j} = \mathbf{J}(\varphi,\nabla\varphi) = -\,\mathbf{a}\,\nabla\varphi \,+\, \mathbf{b}\,\varphi \tag{2.4}$$

and

$$g = q - c\,\varphi \ . \tag{2.5}$$

There thus exists a unique $\varphi \in H^1(\Omega)$ that satisfies the boundary conditions and also (2.3) for all $\mathbf{x}$ in the domain Ω of the problem. This is the **differential formulation**, which is the start point of **finite difference** approximation methods.

The differential equation must be understood in a *weak sense*, i.e.,

$$-\int_\Omega \mathbf{j}\cdot\nabla\psi\ dV + \int_{\partial\Omega}\psi\,\mathbf{j}\cdot\check{\mathbf{n}}\ dS = \int_\Omega g\,\psi\ dV \tag{2.6}$$

for all $\psi \in H^1(\Omega)$. Notice that this formula has no derivative of $\mathbf{j}$ and thus makes sense in cases in which the strong form (2.3) does not.

Considering homogeneous Dirichlet boundary conditions, the **variational formulation** of the problem reads: "Find $\varphi \in H_0^1(\Omega)$ such that

$$-\int_\Omega \mathbf{J}(\varphi,\nabla\varphi)\cdot\nabla\psi\ dV = \int_\Omega g(\varphi)\,\psi\ dV \tag{2.7}$$

for all $\psi \in H_0^1(\Omega)$."

This formulation is adopted in **primal finite element methods**, in which φ_h belongs to some subspace V_h and satisfies (2.7) only for functions ψ belonging to V_h.

Let Γ be a surface that divides Ω into two parts, Ω_1 and Ω_2. Integrating by parts (2.7) in each Ω_i one obtains

$$\int_{\Omega_1} [\nabla\cdot\mathbf{J}(\varphi,\nabla\varphi)-g(\varphi)]\,\psi\,dV+\int_{\Omega_2} [\nabla\cdot\mathbf{J}(\varphi,\nabla\varphi)-g(\varphi)]\,\psi\,dV-\int_{\Gamma} [\![\mathbf{J}(\varphi,\nabla\varphi)\cdot\check{\mathbf{n}}]\!]\,\psi\,dS=0 \qquad \forall\,\psi\,\in\,H_0^1(\Omega)\,. \tag{2.8}$$

This implies that

- The solution of (2.7) satisfied the differential equation a.e. in Ω_1 and Ω_2.
- The normal flux $\mathbf{J}\cdot\check{\mathbf{n}}$ is continuous across Γ.

Exo. 2.1 *Give arguments to support (or prove) both previous statements.*

Let K be an open polyhedral subset of Ω, with facets $e \in \mathcal{E}$. Integrating (2.3) over K and using Gauss-Green formula one gets

$$\sum_{e \in \partial K} \int_e \mathbf{J}(\varphi, \nabla\varphi) \cdot \check{\mathbf{n}} \, dS = \int_K g(\varphi) \, dK \ . \tag{2.9}$$

Notice that $\mathbf{J} \cdot \check{\mathbf{n}}$ is well defined on e. The **integral formulation** of the problem corresponds to "find the unique $\varphi \in H^1(\Omega)$ such that (2.9) holds for all polyhedra K contained in Ω".

- The integral formulation is the basis of **finite volume methods**. The discretization methodology consists of selecting a finite number of polyhedra as the finite volume mesh $\mathcal{T}_h$, and obtaining a finite number of equations by only requiring that (2.9) holds for those polyhedra. This leads to

$$\sum_{e \in \partial K} \overline{F}_{K,e} = \int_K g \, dV \qquad\qquad \forall\, K \in \mathcal{T}_h \ . \tag{2.10}$$

- The next step is the selection of degrees of freedom for the discrete solution. The most usual choice is to have one unknown φ_K per finite volume K, i.e., N_V unknowns for N_V equations. In addition, a node $\mathbf{x}_K$ is defined for each K.

- Letting $\underline{\varphi} \in \mathbb{R}^{N_V}$ be the column array of unknowns, a **numerical flux function** $F_{K,e}(\underline{\varphi})$ is introduced satisfying a **consistency condition**

$$F_{K,e}(\underline{\varphi}^*) \simeq \overline{F}_{K,e}(\varphi, \nabla\varphi) \tag{2.11}$$

  where $\underline{\varphi}^* = (\varphi(\mathbf{x}_1, \varphi(\mathbf{x}_2, \ldots)^T$ is the array of nodal values of any **exact** solution φ of the problem.

- The discrete system of equations reads

$$\sum_{e \in \partial K} F_{K,e}(\underline{\varphi}) = \int_K g(\underline{\varphi}) \, dV \qquad\qquad \forall\, K \in \mathcal{T}_h \ . \tag{2.12}$$

- The finite volume method extends naturally to transient problems. If the equation considered is

$$\partial_t \phi + L\,\phi = q \ , \tag{2.13}$$

  then upon FV discretization in space one ends up with

$$V_K \, \frac{d\varphi_K}{dt} + \sum_{e \in \partial K} F_{K,e}(\underline{\varphi}) = \int_K g(\underline{\varphi}) \; dV \qquad\qquad \forall \, K \, \in \, \mathcal{T}_h \ . \tag{2.14}$$

  Above, V_K is the volume of cell K. The numerical problem thus reduces to a system of ODE that is then discretized in time with a variety of methods.

- For the method to be strictly conservative, it must happen that if a given facet e separates cell K from cell L then

$$F_{K,e}(\underline{\varphi}) = -F_{L,e}(\underline{\varphi}) \ . \tag{2.15}$$

- An interesting alternative to our choice of degrees of freedom is to add an additional unknown per facet. Let $\mathcal{E}$ be the "skeleton" of the mesh, consisting of all facets e, and let $\hat{\varphi}_j$, with $j = 1, \ldots, N_E$ be the facet unknowns. One now has N_V equations and $N_V + N_E$ unknowns. The required additional equations are (2.15), closing the system.

- Other possibilities exist, such as overlapping finite volumes, but we will not discuss them here.

2.2 A one-dimensional example

Let us take

$$L\varphi = -(\mathrm{a}\,\phi_{,1})_{,1} = q \tag{2.16}$$

in the domain $(0, \ell)$, which has nodes $0 = x_0, x_1, \ldots, x_n = \ell$. Let $h_i = x_i - x_{i-1}$. Also, let $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$ and $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$.

Finite differences

$$\begin{aligned}(\mathrm{a}\,\varphi')'(x_j) &\simeq \frac{\mathrm{a}(x_{j+\frac{1}{2}})\varphi'(x_{j+\frac{1}{2}}) - \mathrm{a}(x_{j-\frac{1}{2}})\varphi'(x_{j-\frac{1}{2}})}{h_{j+\frac{1}{2}}} \\ &\simeq \frac{\frac{\mathrm{a}_j+\mathrm{a}_{j+1}}{2}\ \frac{\varphi(x_{j+1})-\varphi(x_j)}{h_{j+1}} - \frac{\mathrm{a}_{j-1}+\mathrm{a}_j}{2}\ \frac{\varphi(x_j)-\varphi(x_{j-1})}{h_j}}{h_{j+\frac{1}{2}}}\ .\end{aligned} \tag{2.17}$$

For equispaced nodes this leads to the discrete scheme (3.9) of Wesseling.

Exo. 2.2 *Build a small code for this problem and solve the interface problem of page 84 of Wesseling. Compare to the results shown in the book.*

Finite volumes

Notice that $\mathrm{J}(\varphi, \varphi') = -\mathrm{a}\,\varphi'$. Letting the finite volumes be given by $V_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ a reasonable numerical flux (for continuous a) is

$$F_{j+\frac{1}{2}} = -\frac{\mathrm{a}_j + \mathrm{a}_{j+1}}{2}\ \frac{\varphi_{j+1} - \varphi_j}{h_{j+1}}\ . \tag{2.18}$$

Exo. 2.3 *Build the corresponding finite volume method in terms of nodal quantities. Compare to the finite-difference scheme.*

Improved finite volumes

Let us introduce as additional degrees of freedom the values $\varphi_{j+\frac{1}{2}}$ and

$$F_{j,j+\frac{1}{2}} = -\,\mathrm{a}_j\,\frac{\varphi_{j+\frac{1}{2}} - \varphi_j}{h_{j+1}/2}\ . \tag{2.19}$$

Similarly, we have

$$F_{j+1,j+\frac{1}{2}} = \mathrm{a}_{j+1}\,\frac{\varphi_{j+1} - \varphi_{j+\frac{1}{2}}}{h_{j+1}/2}\ . \tag{2.20}$$

Conservation condition (2.15) then allows to eliminate the unknown $\varphi_{j+\frac{1}{2}}$,

$$F_{j+\frac{1}{2}} = F_{j,j+\frac{1}{2}} = -F_{j+1,j+\frac{1}{2}} \qquad \Rightarrow \qquad \varphi_{j+\frac{1}{2}} = \frac{\mathrm{a}_j\varphi_j + \mathrm{a}_{j+1}\varphi_{j+1}}{\mathrm{a}_j + \mathrm{a}_{j+1}}\ . \tag{2.21}$$

Exo. 2.4 *Build the finite volume scheme corresponding to the flux above. Compare to (3.17) de Wesseling. Modify the code of exercise 2.2 to implement it. Test it. Compute the convergence order in a smooth problem with analytical solution.*

Exo. 2.5 *Study and discuss cell-centered finite volumes for the 1D problem, in which the nodes are* $x_{j+\frac{1}{2}}$ *instead of* x_j *and the finite volumes are of the form* (x_j, x_{j+1})*. Modify the code to deal with cell-centered discretization and compare to previous results.*

Exo. 2.6 *Analyze the consistency (truncation error) of the fluxes and of the overall stencil of the vertex-centered scheme of Exo. 2.3. Consider* $\mathrm{a} \equiv 1$*,* $f = 1$ *and* h_i *equal to* h *if* i *is even and equal to* $h/2$ *when* i *is odd. Discuss the result together with a numerical experiment.*

Exo. 2.7 *Study Chapter 2 (and part of Chapter 3) of Eymard et al's "Finite Volume Methods":*

1. *What is the definition of an admissible one-dimensional mesh? Are cell-centered and vertex-centered meshes admissible?*

2. *Do the calculations showing that a cell-centered scheme is not consistent in the usual finite-difference sense (Example 2.1 and Remark 2.3).*

3. *Follow step by step the proof of Theorem 2.1.*

4. *Do the calculations that lead to equation 2.26 and to the harmonic mean formula of Example 2.2.*

5. *Follow step by step the proof of Theorem 2.3.*

6. *What is an admissible mesh in 2D? Give examples of admissible meshes and of inadmissible meshes. What is a Voronoïmesh? Are Voronoïmeshes always admissible? Why?*

7. *Explain Equation 3.86: What is the equation corresponding to a cell that has its boundary at the boundary of* Ω*?*

8. *Explain in simple words Definition 3.7 of Neumann restricted admissible meshes.*

Exo. 2.8 *(***Miniproject***) Consider a microchannel with (electrically) non-conducting walls and a conducting fluid. The geometry is given by a mask on a rectangular mesh, so that the mask takes different values depending on the cell being fluid, wall, inlet, outlet. The electric potential satisfies*

$$\Delta\Phi = 0 \tag{2.22}$$

in the fluid, with $\partial\Phi/\partial n = 0$ *at the walls, and* Φ *given at inlets and outlets. Code a* finite volume *solver for the electric potential and compute from it the electric field* $\mathbf{E} = -\nabla\Phi$.
In electro-osmotic flows *with homogeneous material properties the fluid velocity satisfies*

$$\mathbf{u} = -\,\kappa\,\mathbf{E} \tag{2.23}$$

where κ *is a material constant. With the computed electric field simulate the transport of inert particles for a non-straight microchannel.*
Also, take a look at the video

`https://br.comsol.com/video/simulating-electrokinetic-phenomena-microfluidics#`

3 Numerical approximation of fully developed flow

3.1 The physical setting

- Incompressible flow along a long cylinder of cross section $\Omega \subset \mathbb{R}^2$. The flow domain is $\mathcal{B} = \Omega \times (0, L)$.
- The flow is driven by a pressure gradient

$$\mathcal{G} = \frac{p(L) - p(0)}{L} \tag{3.1}$$

  notice that when $\mathcal{G} > 0$ we expect $w = u_3 < 0$ and viceversa.

- If L is sufficiently large, the entry and exit effects can be neglected and all cross sections are essentially identical, except for the pressure.
- Decomposing the stress tensor in pressure and non-pressure components, we assume

$$\boldsymbol{\sigma}(x_1, x_2, x_3, t) = -p(x_3, t)\,\mathbb{I} + \boldsymbol{\sigma}^*(x_1, x_2, t)\ . \tag{3.2}$$

- Let ω be an arbitrary region in Ω and let V be the corresponding cylinder, i.e.,

$$V = \omega \times (0, L)\ . \tag{3.3}$$

  We denote also $\omega_z = \omega \times \{z\}$ (the cross section at $x_3 = z$) and $\mathcal{S} = \partial\omega \times (0, L)$ (the lateral surface) so that

$$\partial V = \omega_0 \cup \mathcal{S} \cup \omega_L\ . \tag{3.4}$$

3.2 Conservation principles

- Mass: Because of incompressibility, and assuming ρ is a constant, this principle reads

$$0 = \int_{\partial V} \mathbf{u} \cdot \check{\mathbf{n}} \, dS = - \int_{\omega_0} w \, dS \; + \; \int_{\omega_L} w \, dS \; + \; \int_{\mathcal{S}} \mathbf{u} \cdot \check{\mathbf{n}} \, dS \; . \tag{3.5}$$

  This condition is automatically satisfied in *parallel flows* which we consider hereafter, i.e., flows in which the velocity is of the form

$$\mathbf{u}(x_1, x_2, x_3, t) = (0, 0, w(x_1, x_2, t)) \; . \tag{3.6}$$

- Momentum: In parallel flows,

$$L \, \frac{d}{dt} \int_{\omega} \rho \, w \, d\omega = -\mathcal{G} \, L \, |\omega| \; + L \int_{\partial\omega} \boldsymbol{\tau} \cdot \check{\boldsymbol{\nu}} \, d\partial\omega \tag{3.7}$$

  where

$$\underline{\boldsymbol{\tau}} = (\sigma_{13}, \sigma_{23})^T \qquad \text{and} \qquad \underline{\check{\boldsymbol{\nu}}} = (\mathrm{n}_1, \mathrm{n}_2)^T \; . \tag{3.8}$$

In incompressible isothermal flows the mass and momentum conservation principles form a closed system.

In this case one equation, which is (3.7), in one unknown w.

3.3 Boundary conditions

Exo. 3.1 *Read Section 1.6 and Chapter 2 of Kirby.*

1. *Prove Eq. 1.59.*

2. *What is the Navier slip boundary condition? Give a physical argument that determines the sign of* b.

3. *Solve the unidirectional flow between two parallel plates located at* $x_3 = 0$ *and* $x_3 = h$, *subject to a pressure gradient* $\nabla p = (\partial_1 p, \partial_2 p)$ *and with the upper plate moving at a velocity* $\mathbf{u} = (U, 0)$ *with respect to the lower one. Compute the flux* $\mathbf{j} = \int_0^h \mathbf{u}\, dx_3$ *as a function of* ∇p, U *and* h *(and of the fluid viscosity* μ*).*

4. *Justify the claim* $\nabla \cdot \mathbf{j} = \partial_1 j_1 + \partial_2 j_2 = 0$, *and use this claim to arrive at the* **lubrication equation** *(also known as Reynolds equation).*

- The **no-slip boundary condition** holds when a fluid is in contact with a solid surface, in this case it translates to

$$w(x_1, x_2, t) = 0 \qquad \forall\,(x_1, x_2) \in \partial\Omega\ . \tag{3.9}$$

- Under certain conditions, the fluid has been observed to **slip** at the solid boundary (e.g., in very rarefied flows). In the parallel flows we are considering, the adopted (Navier) condition amounts to

$$\boldsymbol{\tau} \cdot \check{\boldsymbol{\nu}} = -\,b\,(w - w_{\text{wall}})\ . \tag{3.10}$$

- If an electric field is applied along the channel, then a non-zero velocity "appears" at the wall (read Chapter 6 of Kirby to understand why). This is an apparent wall velocity, which in fact only takes place at a finite distance $\sim\ 5\,\lambda_D$ from it, where λ_D is the **Debye-Hückel length**. Its value is given by the **Helmholtz-Smoluchowski** equation

$$w = m_{eo}\,E_{\text{wall}} \tag{3.11}$$

  where m_{eo} is the **electroosmotic mobility** and is a property of the fluid and the surface material (of the order of $10^{-8} m^2/(V\,s)$).

3.4 Viscous parallel flow

If the fluid is Newtonian-like (Boussinesq),

$$\boldsymbol{\sigma}^* = \mu \begin{pmatrix} 0 & 0 & w_{,1} \\ 0 & 0 & w_{,2} \\ w_{,1} & w_{,2} & 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{\tau} = \mu \nabla w \ . \tag{3.12}$$

We can, applying Gauss-Green theorem, rewrite (3.7) as

$$\int_\omega \left[\rho \, \partial_t \, w + \mathcal{G} - \nabla \cdot (\mu \, \nabla w)\right] \, d\omega = 0 \tag{3.13}$$

and arrive at the differential form (with no-slip conditions for example)

$$\begin{cases} \rho \, \partial_t \, w + \mathcal{G}(t) - \nabla \cdot (\mu \, \nabla w) = 0 & \text{in } \Omega \ , \\ w = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{3.14}$$

Writing it as a conservation law

$$\partial_t(\rho \, w) + \nabla \cdot \mathbf{j} = g \ , \qquad \mathbf{j} = -\mu \nabla w \ , \qquad g = -\mathcal{G} \ . \tag{3.15}$$

3.5 Discretization in Cartesian grids

3.5.1 Finite differences

Consider a rectangular pipe $\Omega = (0, L_1) \times (0, L_2)$ with a uniform vertex-centered Cartesian grid with nodes at positions

$$\mathbf{X}_{j_1 j_2} = ((j_1 - 1)h_1, (j_2 - 1)h_2), \qquad j_\alpha = 1, \ldots, n_\alpha + 1, \qquad \alpha \in \{1, 2\} \ , \tag{3.16}$$

where n_α is the number of subdivisions in the α direction and $n_\alpha h_\alpha = L_\alpha$.

Considering as unknowns the values at the nodes w_{j_1,j_2}, we have $w_{j_1,j_2} = 0$ if (j_1, j_2) is at the boundary. For an internal node, on the other hand, a FD space discretization of (3.14) with constant density and viscosity leads to

$$\rho \, \frac{d}{dt} w_{j_1,j_2} + \mathcal{G} - \mu \, \frac{w_{j_1+1,j_2} - 2\, w_{j_1,j_2} + w_{j_1-1,j_2}}{h_1^2} - \mu \, \frac{w_{j_1,j_2+1} - 2\, w_{j_1,j_2} + w_{j_1,j_2-1}}{h_2^2} = 0 \ . \tag{3.17}$$

Our first issue is the implementation of this method.

Node-to-unknown mapping:

There are $(n_1+1)\times(n_2+1)$ unknowns, they can be numbered by row or by column (or else) to get the mapping. Denoting $N_1 = n_1+1$, $N_2 = n_2+1$,

```
function ng = ij2n (i,j)
  global N1 N2
  ng = i + (j-1)*N1;
endfunction
```

Exo. 3.2 *Build a function* `n2ij(n)` *that is the inverse of the previous one.*

Viscous matrix:

```
pP=ij2n(i,j); pN=ij2n(i,j+1); pE=ij2n(i+1,j); pS=ij2n(i,j-1); pW=ij2n(i-1,j);
```

The following matrix row provides the viscous contribution $(L_\mu w)_P \simeq -\mu\,\nabla^2 w(P)$ to equation P (interior node):

```
aux1 = mu/dx^2; aux2 = mu/dy^2;
A(pP,pP) = 2*(aux1+aux2);
A(pP,pN)=-aux2; A(pP,pS)=-aux2;
A(pP,pE)=-aux1; A(pP,pW)=-aux1;
```

so that

$$-\,\mu\,\frac{w_{j_1+1,j_2}-2\,w_{j_1,j_2}+w_{j_1-1,j_2}}{h_1^2}-\mu\,\frac{w_{j_1,j_2+1}-2\,w_{j_1,j_2}+w_{j_1,j_2-1}}{h_2^2}=\left(\underline{\underline{A}}\;\underline{W}\right)_P\quad . \tag{3.18}$$

Considering just the interior nodes, we get the system

$$\rho\,\frac{d}{dt}\underline{W}\;+\;\underline{\underline{A}}\;\underline{W}\;=\;\underline{b}(t) \tag{3.19}$$

where $b_P(t) = -G(t)$. Discretizing now in time by the θ-method,

$$\left(\frac{\rho}{\Delta t}\,\underline{\underline{\mathbf{I}}}+\;\theta\,\underline{\underline{A}}\right)\,\underline{W}^{n+1}=\left(\frac{\rho}{\Delta t}\,\underline{\underline{\mathbf{I}}}-\;(1-\theta)\,\underline{\underline{A}}\right)\,\underline{W}^{n}+\;\underline{b}^{n+\theta} \tag{3.20}$$

or

$$\underline{\underline{M}}\,\underline{W}^{n+1}=\;\underline{\underline{R}}\,\underline{W}^{n}+\;\underline{b}^{n+\theta} \tag{3.21}$$

```
#-- Assembly: loop over nodes
for i=1:N1
   for j=1:N2
      if (i==1 || i==N1 || j==1 || j==N2)
         continue;
      else
# viscous matrix
         pP=ij2n(i,j); pN=ij2n(i,j+1); pE=ij2n(i+1,j); pS=ij2n(i,j-1); pW=ij2n(i-1,j);
         aux1 = mu/dx^2; aux2 = mu/dy^2;
         Af(pP,pP) = 2*(aux1+aux2);
         Af(pP,pN)=-aux2; Af(pP,pS)=-aux2; Af(pP,pE)=-aux1; Af(pP,pW)=-aux1;
# mass matrix
         Am(pP,pP)=rho/dt;  bm(pP)=dx*dy;
      endif
   endfor
endfor
#-- Timestepping Matrices: M, R
M = Am + theta*Af;
R = Am - (1-theta)*Af;
#-- Correct M for no-slip boundary conditions
for i=1:N1
   for j=1:N2
      if (i==1 || i==N1 || j==1 || j==N2)
         pP=ij2n(i,j); M(pP,pP)=1;
      endif
   endfor
endfor
```

Exo. 3.3 (Miniproject: electroosmotic pump) *Consider a pipe of rectangular cross section* $(0, W) \times (0, H)$ *and length* L*, such that the horizontal walls are made of glass and the vertical ones of PDMS. Considering water as the fluid, the corresponding electroosmotic mobilities are* $m_{eo} = 3 \times 10^{-8}$ *and* 1.5×10^{-8} $m^2/(V\text{-}s)$*. The water has* $\rho = 1000$ kg/m^3 *and* $\mu = 10^{-3}$ $Pa\text{-}s$*.*
Take $W = 20$ *microns,* $H = 10$ *microns and* $L = 3$ *mm.*
Adapt the code `pipe_fd_t.m` *and answer the following questions:*

1. *Considering that both ends of the pipe are at atmospheric pressure, what is the steady flow rate and average velocity for a voltage difference of 1 Volt between the ends of the pipe? What is the shape of the steady velocity profile? How long does it take to reach the steady flow rate?*

2. *If one end of the pipe is closed, what will be the pressure difference between its ends?*

3.6 Vertex-centered finite volumes

- The node-to-unknown mapping remains the same. To allow for variable spacing we assume that arrays $X(1:N_1)$ and $Y(1:N_2)$ are given, containing the nodal coordinates.

- From (3.7), the equation for the (interior) finite volume P is

$$F_{PN} + F_{PE} + F_{PS} + F_{PW} = \int_{\omega_P} (-\mathcal{G} - \rho\, \partial_t w)\; d\omega \simeq m(\omega_P) \left(-\mathcal{G} - \rho\, \frac{dW_P}{dt} \right) \tag{3.22}$$

  where we have treated $\partial_t w$ as a source and the left-hand side approximates $\int_{\partial\omega_P} \mathbf{j} \cdot \check{\boldsymbol{\nu}}\; ds$ (remember that $\mathbf{j} = -\mu \nabla w$).

- Now we have to define the discrete fluxes

$$\int_{e_N} \mathbf{j} \cdot \check{\boldsymbol{\nu}}\; dx_1 = \int_{e_N} \mathrm{j}_2\; dx_1 = \int_{e_N} (-\mu\, w_{,2})\; dx_1 \simeq -\mu\, \frac{W_P - W_N}{y_P - y_N}\, \frac{x_E - x_W}{2} \doteq F_{PN} \tag{3.23}$$

  and analogously

$$F_{PE} \doteq -\mu\, \frac{W_P - W_E}{x_P - x_E}\, \frac{y_N - y_S}{2} \tag{3.24}$$

$$F_{PS} \doteq \mu\, \frac{W_P - W_S}{y_P - y_S}\, \frac{x_E - x_W}{2} \tag{3.25}$$

$$F_{PW} \doteq \mu\, \frac{W_P - W_W}{x_P - x_W}\, \frac{y_N - y_S}{2} \tag{3.26}$$

- If we consider the mesh uniform and divide everything by $h_1 h_2$, we arrive at the discrete equation

$$\rho \frac{dW_P}{dt} + \mu \frac{W_P - W_N}{h_2^2} + \mu \frac{W_P - W_S}{h_2^2} + \mu \frac{W_P - W_E}{h_1^2} + \mu \frac{W_P - W_W}{h_1^2} = -\mathcal{G} \ , \tag{3.27}$$

  which shows that balancing fluxes over control volumes indeed leads to a discretization of the Laplacian (equivalent to finite differences, in simple cases).

- In the FV case the inertia matrix is diagonal but not proportional to the identity:

$$B_{PP} = m(\omega_P)\,\rho\ . \tag{3.28}$$

- Similarly, the right-hand side is now

$$b_P = -\,m(\omega_P)\,\mathcal{G}\ . \tag{3.29}$$

- The viscous matrix can be built by summing up the contributions of each face:

```
# viscous matrix
    pP=ij2n(i,j); pN=ij2n(i,j+1); pE=ij2n(i+1,j); pS=ij2n(i,j-1); pW=ij2n(i-1,j);
    xP=X(i); xN=xP; xS=xP; xE=X(i+1); xW=X(i-1);
    yP=Y(j); yE=yP; yW=yP; yN=Y(j+1); yS=Y(j-1);
# north face
    aux=mu*(xE-xW)/(yN-yP)/2;
    Af(pP,pP) = Af(pP,pP) + aux;
    Af(pP,pN) = Af(pP,pN) - aux;
# east face
    aux=mu*(yN-yS)/(xE-xP)/2;
    Af(pP,pP) = Af(pP,pP) + aux;
    Af(pP,pE) = Af(pP,pE) - aux;
etcetera
```

- Variable viscosity: Let us assume that the viscosity is not uniform, but a given function $\mu(x, y)$. The modification in the previous code is straightforward:

```
# viscous matrix
    pP=ij2n(i,j); pN=ij2n(i,j+1); pE=ij2n(i+1,j); pS=ij2n(i,j-1); pW=ij2n(i-1,j);
    xP=X(i); xN=xP; xS=xP; xE=X(i+1); xW=X(i-1);
    yP=Y(j); yE=yP; yW=yP; yN=Y(j+1); yS=Y(j-1);
# north face
    xF=xP; yF=(yP+yN)/2;
    aux=mu(xF,yF)*(xE-xW)/(yN-yP)/2;
    Af(pP,pP) = Af(pP,pP) + aux;
    Af(pP,pN) = Af(pP,pN) - aux;
# east face
    xF=(xP+xE)/2; yF=yP;
    aux=mu(xF,yF)*(yN-yS)/(xE-xP)/2;
    Af(pP,pP) = Af(pP,pP) + aux;
    Af(pP,pE) = Af(pP,pE) - aux;
etcetera
```

With the FV formulation we arrive at the system

$$\underline{\underline{B}}\,\frac{d}{dt}\underline{W} \;+\; \underline{\underline{A}}\,\underline{W} \;=\; \underline{b}(t)\;. \tag{3.30}$$

Discretizing now in time by the θ-method,

$$\left(\frac{1}{\Delta t}\,\underline{\underline{B}} + \,\theta\,\underline{\underline{A}}\right)\,\underline{W}^{n+1} = \left(\frac{1}{\Delta t}\,\underline{\underline{B}} - \,(1-\theta)\,\underline{\underline{A}}\right)\,\underline{W}^{n} + \,\underline{b}^{n+\theta} \tag{3.31}$$

or

$$\underline{\underline{M}}\,\underline{W}^{n+1} = \underline{\underline{R}}\,\underline{W}^{n} + \,\underline{b}^{n+\theta}\;. \tag{3.32}$$

Quite similar to the FD, uniform spacing case, but now with more general properties and mesh.

- Quasi-newtonian fluid: Viscosity may depend on the shear rate, for incompressible flows given by

$$\dot{\gamma} \doteq \sqrt{D\,\mathbf{u} : D\,\mathbf{u}} \tag{3.33}$$

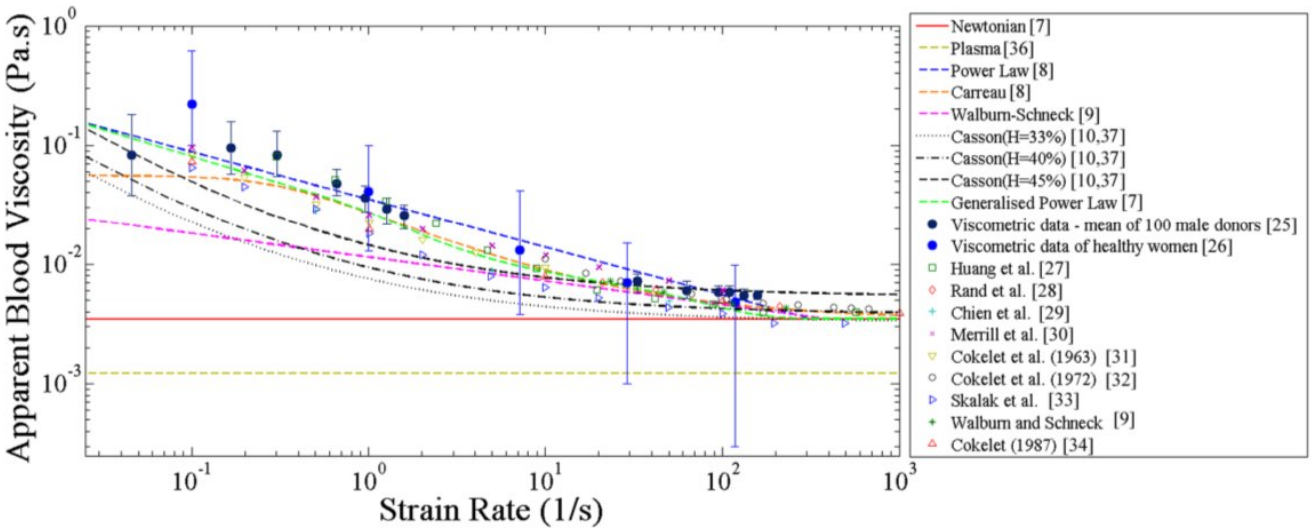


Fig 2. Experimental measurements of blood viscosity and non-Newtonian blood rheological models.

doi:10.1371/journal.pone.0128178.g002

Different models exist for blood

Table 1. Blood rheological model equations.

Blood Model	Effective Viscosity (Pa·s)
Newtonian [7]	$\mu = 0.00345\ Pa{\cdot}s$
Plasma [36]	$\mu = 0.00122\ Pa{\cdot}s$
Power Law (Modified) [8]	$\mu = \begin{cases} m(\dot{\gamma})^{n_P-1}, & \dot{\gamma} < 427 \\ 0.00345\,Pa \cdot s, & \dot{\gamma} \geq 427 \end{cases}$, $m = 0.035$, $n_P = 0.6$
Walburn-Schneck (Modified) [9]	$\mu = \begin{cases} C_1 e^{(C_2 H)} e^{\left(C_4\left(\frac{TPMA}{H^2}\right)\right)} (\dot{\gamma})^{-C_3 H}, & \dot{\gamma} < 414 \\ 0.00345\,Pa \cdot s, & \dot{\gamma} \geq 414 \end{cases}$, C1 = 0.00797, C2 = 0.0608, C3 = 0.00499, C4 = 14.585, H = 40, TPMA = 25.9
Casson [10,37]	$\mu = 0.1\left(\left[\sqrt{\eta} + \sqrt{\tau_y\left(\frac{1-e^{-m\lvert\dot{\gamma}\rvert}}{\lvert\dot{\gamma}\rvert}\right)}\right]^2\right)$, $\tau_y = (0.625H)3$, $\eta = \eta 0(1\text{-}H)^{-2.5}$, $\eta_0 = 0.012$, H = 40% (female normal), 33% (post-angioplasty) or 45% (male normal)
Carreau [8]	$\mu = \mu_{\infty C} + (\mu_0 - \mu_{\infty C})[1 + (\lambda\dot{\gamma})^2]^{\frac{n_C-1}{2}}$, $\lambda = 3.313$, $n_C = 0.3568$, $\mu_0 = 0.056$, and $\mu_{\infty C} = 0.00345$
Generalised Power Law [7]	$\mu = \lambda\lvert\dot{\gamma}\rvert^{n-1}, \lambda = \mu_{\infty G} + \Delta\mu exp\left[-\left(1 + \frac{\lvert\dot{\gamma}\rvert}{a}\right)exp\left(-\frac{b}{\lvert\dot{\gamma}\rvert}\right)\right]$, $n = n_\infty - \Delta n\, exp\left[-\left(1 + \frac{\lvert\dot{\gamma}\rvert}{c}\right)exp\left(-\frac{d}{\lvert\dot{\gamma}\rvert}\right)\right]$, $\mu_{\infty G} = 0.0035$, $n_\infty = 1.0$, $\Delta\mu = 0.025$, $\Delta n = 0.45$, a = 50, b = 3, c = 50, and d = 4

doi:10.1371/journal.pone.0128178.t001

3.7 Cell-centered finite volumes

The same problems as before can be solved by cell-centered finite volumes. It is interesting to see how the imposition of boundary conditions is quite different.
We adopt a convention for the numeration of unknowns in structured quadrilateral finite volumes.

- We consider a "covering domain" $(x^-, x^+) \times (y^-, y^+)$. The mesh is provided by two arrays, X and Y, such that

$$x^- = X_1 < X_2 < \ldots < X_{n_1+1} = x^+ \ , y^- = Y_1 < Y_2 < \ldots < Y_{n_2+1} = y^+ \ . \tag{3.34}$$

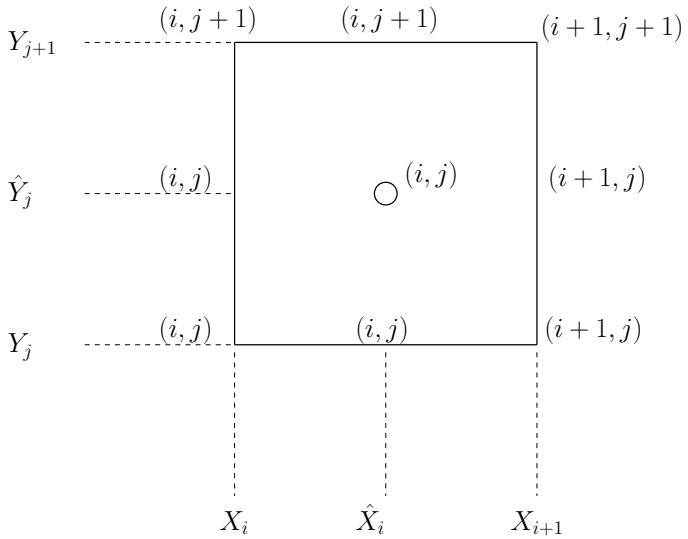
- The **cell** with numbering (i, j) will be

$$V_{ij} = (X_i, X_{i+1}) \times (Y_j, Y_{j+1}) \ . \tag{3.35}$$

- The **cell unknowns** have the same numbering as the corresponding cell, and it is located at the **nodes** given by the arrays $\hat{X}$ and $\hat{Y}$:

$$\begin{aligned} W_{ij} &\simeq w(\hat{X}_i, \hat{Y}_j) && (3.36)\\ \hat{X}_i &= \frac{1}{2}(X_i + X_{i+1}) && (3.37)\\ \hat{Y}_j &= \frac{1}{2}(Y_j + Y_{j+1}) && (3.38)\\ \delta x_i &= X_{i+1} - X_i && (3.39)\\ \delta y_j &= Y_{j+1} - Y_j && (3.40) \end{aligned}$$

- The face and vertex unknowns will be numbered as follows:



- All cells will have all unknowns, meaning that there will be:
  - $n_1 \times n_2$ cell unknowns.
  - $(n_1 + 1) \times (n_2 + 1)$ vertex unknowns.
  - $n_1 \times (n_2 + 1)$ horizontal face unknowns.
  - $(n_1 + 1) \times n_2$ vertical face unknowns.
- A **mask** will be a **cell variable** M_{ij}, with $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, such that if $M_{ij} = 0$ we have a **fluid cell**. Other values of the mask will correspond to **walls**, which can have different boundary conditions depending on the value.

Exo. 3.4 Miniproject (electroosmotic pump 2): *Build a cell-centered code for the solution of the transient parallel flow of a viscous fluid with the following characteristics:*

- *The pressure gradient $\mathcal{G}(t)$ can be arbitrary.*
- *Time integration is performed with the method of lines (θ-method).*
- $M_{ij} = 11, 12 \quad \Rightarrow \quad$ *Smoluchowski condition, with m_{eo} given and $E_{wall}(t)$ programmable.*
- *Computes the flow rate $Q = \int_\Omega w \, d\Omega$ as a function of time.*

Then answer the same questions as in the previous miniproject.

4 Incompressible Navier-Stokes equations

- In the previous sections we have applied the basic principles of fluid mechanics to *parallel flows*.
- In more general situations, one has to go back to the basic principles as introduced in section 1. They can be equivalently written in integral or differential form, and the latter can equivalently be conservative or non-conservative.
- In this section we particularize the basic principles for the case of an incompressible Newtonian fluid, arriving at the Navier-Stokes equations.

4.1 Equations and fluxes

- The basic conservation principles of Continuum Mechanics, as discussed in the first chapter, are **conservation of mass**,

$$\frac{d}{dt}\int_\Omega \rho\, d\Omega + \int_{\partial\Omega} \rho\,\mathbf{u}\cdot\check{\mathbf{n}}\; d\partial\Omega = 0\ , \tag{4.1}$$

conservation of momentum,

$$\frac{d}{dt}\int_\Omega \rho\,\mathbf{u}\, d\Omega + \int_{\partial\Omega} (\rho\,\mathbf{u}\otimes\mathbf{u} - \boldsymbol{\sigma})\cdot\check{\mathbf{n}}\; d\partial\Omega = \int_\Omega \mathbf{f}\; d\Omega\ , \tag{4.2}$$

conservation of energy,

$$\frac{d}{dt}\int_\Omega E\, d\Omega + \int_{\partial\Omega} (E\,\mathbf{u} - \boldsymbol{\sigma}\cdot\mathbf{u} + \mathbf{q})\cdot\check{\mathbf{n}}\; d\partial\Omega = \int_\Omega (\mathbf{f}\cdot\mathbf{u} + Q)\; d\Omega\ , \tag{4.3}$$

where the fourth principle, **conservation of angular momentum**, is automatically satisfied by requiring that the **stress tensor** $\boldsymbol{\sigma}$ be symmetric. Above, E is the **total energy**, i.e. $E = \rho\,(e + \mathbf{u}\cdot\mathbf{u}/2)$, and $\mathbf{q}$ is the **heat flux**.

- The **fluxes** of mass, momentum and energy are highlighted in **red** above.

- A **Newtonian fluid** is defined by the linear **constitutive relation**

$$\boldsymbol{\sigma} = -\,p\,\mathbb{I} + \mu\,\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \lambda\,\nabla\cdot\mathbf{u}\,\mathbb{I}\,. \tag{4.4}$$

- Under some frequent conditions ρ is constant and μ can be assumed given, which leads to a **closed system of equations** which only involves the mass and momentum conservation principles.

Exo. 4.1 *Deduce the following equations and write them down explicitly in 2D Cartesian coordinates.*

Incompressible Navier-Stokes equations (non-conservative form):

$$\begin{aligned} \rho\,\partial_t\mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\;\mathbf{u} - \nabla\;\cdot\;\left[\mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right] + \nabla p &= \mathbf{f} && (4.5)\\ \nabla\cdot\mathbf{u} &= 0 && (4.6) \end{aligned}$$

The simplest form, when μ is constant, reads

$$\begin{aligned} \rho\,\partial_t\mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\;\mathbf{u} - \mu\,\nabla^2\mathbf{u} + \nabla p &= \mathbf{f} && (4.7)\\ \nabla\cdot\mathbf{u} &= 0 && (4.8) \end{aligned}$$

Exo. 4.2 *Prove that for an incompressible flow all the following expressions for the acceleration are equivalent:*

$$\frac{D\mathbf{u}}{Dt} = \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \partial_t\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) = \partial_t\mathbf{u} - \mathbf{u}\times(\nabla\times\mathbf{u}) + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) = \partial_t\mathbf{u} + \frac{1}{2}\left[(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u})\right] \tag{4.9}$$

They are called **convective**, **conservative**, **rotational** *and* **skew-symmetric** *forms, respectively.*

Exo. 4.3 *Deduce that the differential equations in conservative form are the*

Incompressible Navier-Stokes equations (conservative form):

$$\rho\,\partial_t \mathbf{u} + \nabla\cdot\left[p\,\mathbb{I} - \mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \rho\,\mathbf{u}\otimes\mathbf{u}\right] = \mathbf{f} \tag{4.10}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{4.11}$$

Incompressible Navier-Stokes equations (integral form):

$$\int_V \rho\,\partial_t\mathbf{u}\;dV + \int_S \boldsymbol{\zeta}\cdot\check{\mathbf{n}}\;dS = \int_V \mathbf{f}\;dV \tag{4.12}$$

$$\int_S \mathbf{u}\cdot\check{\mathbf{n}}\;dS = 0 \tag{4.13}$$

where the momentum flux $\boldsymbol{\zeta}$ is

$$\boldsymbol{\zeta} = p\,\mathbb{I} - \mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \rho\,\mathbf{u}\otimes\mathbf{u} = -\boldsymbol{\sigma} + \rho\,\mathbf{u}\otimes\mathbf{u}\;. \tag{4.14}$$

- The main difficulties for the numerical simulation of incompressible flows come from
  1. The **coupling of velocity and pressure**.
  2. The **nonlinearity of the convective term**.
  3. If viscosity is small (Re_h large), the convective term **dominates** the viscous term (singular perturbation, boundary layers, etc.).

4.2 Initial and boundary conditions

- The equations (4.5)-(4.6) are first order in time and second order in space for the velocity. The well-posedness of the problem requires **an initial velocity field**, which has to be **divergence-free**, i.e.,

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}) \qquad with \nabla \cdot \mathbf{u}_0 = 0 \qquad in \Omega \ . \tag{4.15}$$

  Remark: In fact, it is also required that the **imposed normal velocity** *at the boundary be compatible with* $\mathbf{u}_0$.

- There are several types of possible boundary conditions:

  1. **Imposed velocity:** At rigid walls, if $\mathbf{u}_w$ is the velocity of the wall, set $\mathbf{u} = \mathbf{u}_w$.
  2. **Imposed force:** Used when the force $\mathbf{L}$ applied on the fluid (per unit surface) at some boundary is known. The condition reads

$$\boldsymbol{\sigma} \cdot \check{\mathbf{n}} = \mathbf{L} \ . \tag{4.16}$$

  3. **Drag law:** This corresponds to

$$\boldsymbol{\sigma} \cdot \check{\mathbf{n}} = -\mathbf{D}(\mathbf{u}) \ . \tag{4.17}$$

     An impermeable wall with drag would have the following condition:

$$\mathbf{u} \cdot \check{\mathbf{n}} = 0 \ , \qquad (\boldsymbol{\sigma} \cdot \check{\mathbf{n}})_\tau = -\mathbf{D}(\mathbf{u}) \ , \tag{4.18}$$

     where $\mathbf{v}_\tau$ refers to the tangential component of a vector $\mathbf{v}$, i.e.,

$$\mathbf{v}_\tau = \mathbf{v} \ - \ (\mathbf{v} \cdot \check{\mathbf{n}})\, \check{\mathbf{n}} \ . \tag{4.19}$$

  4. The **drag law** is an example of the decomposition between **tangential** and **normal** conditions at boundaries. Typically, one specifies the tangential component of either velocity or force, and the normal component of either velocity or force, with these two choices being independent of one another.

5. **Free surface with surface tension:**

$$\boldsymbol{\sigma} \cdot \check{\mathbf{n}} = -\gamma H \check{\mathbf{n}} + (\nabla \gamma)_\tau \tag{4.20}$$

   where H is the mean curvature.

6. **Outflow:** Some combination of the above that tries to minimize the upstream effect of domain truncation.

The incompressible Navier-Stokes equations are **by themselves** widely used in Physics and Engineering, thus justify the interest in their numerical approximation.

They are also a **fundamental building block** of more sophisticated models that can predict the behavior of **thermally buoyant flows**, **averaged turbulent flows**, **two-phase flows**, among others.

5 The MAC discretization method of the incompressible Navier-Stokes equations

5.1 Conservation in a rectangle

Consider a finite volume V which is a rectangle of sides h_x and h_y, and denote its edges by E, W, N and S, with exterior normals $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$, respectively. Our aim here is to obtain explicit expressions for the mass and momentum conservation equations in this rectangle.
The mass flux vector $\rho\,\mathbf{u}$ must satisfy

$$\int_{\partial V} \rho\,\mathbf{u}\cdot\check{\mathbf{n}}\;dS = 0\;. \tag{5.1}$$

The momentum flux vector

$$\boldsymbol{\zeta} = p\,\mathbb{I} - \mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \rho\,\mathbf{u}\otimes\mathbf{u} \tag{5.2}$$

consists of three terms, which we denote by *pressure*, *viscous* and *inertia* terms.
The momentum equation contains the integral of $\boldsymbol{\zeta}\cdot\check{\mathbf{n}}$ over the boundary of V, which is the only nontrivial part to calculate and is detailed below. In what follows we adopt the usual notation of (x,y) instead of (x_1,x_2), and (u,v) instead of (u_1,u_2).

5.2 Mass conservation

The exact mass conservation equation reads

$$(u_E^* - u_W^*)\, h_y + (v_N^* - v_S^*)\, h_x = 0 \ . \tag{5.3}$$

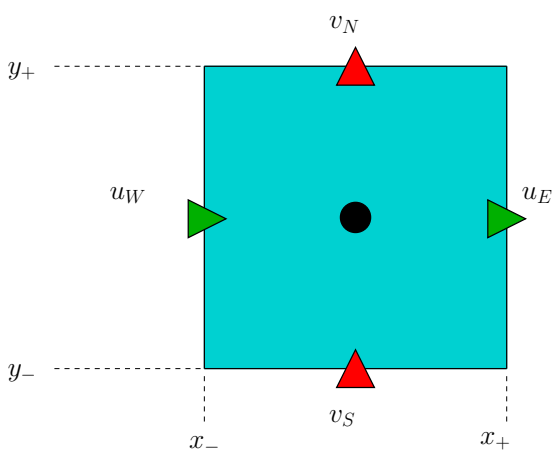
where u_E^*, u_W^*, v_N^* and v_S^* are the face-averaged velocities over the East, West, North and South faces; i.e.,

$$u_E^* = \frac{1}{h_y}\int_E \mathbf{u}\cdot\check{\mathbf{n}}\; ds = \frac{1}{h_y}\int_{y_-}^{y_+} u(x_E, y)\; dy \tag{5.4}$$

$$u_W^* = -\frac{1}{h_y}\int_W \mathbf{u}\cdot\check{\mathbf{n}}\; ds = \frac{1}{h_y}\int_{y_-}^{y_+} u(x_W, y)\; dy \tag{5.5}$$

$$v_N^* = \frac{1}{h_x}\int_N \mathbf{u}\cdot\check{\mathbf{n}}\; ds = \frac{1}{h_x}\int_{x_-}^{x_+} v(x, y_N)\; dx \tag{5.6}$$

$$v_S^* = -\frac{1}{h_x}\int_S \mathbf{u}\cdot\check{\mathbf{n}}\; ds = \frac{1}{h_x}\int_{x_-}^{x_+} v(x, y_S)\; dx \tag{5.7}$$



Basic methodology:

1. Define **degrees of freedom** for $\mathbf{u}$: $\theta_1(\mathbf{u})$, $\theta_2(\mathbf{u})$, etc.
2. Approximate $u_E \simeq u_E^*$, $u_W \simeq u_W^*$, $v_N \simeq v_N^*$ and $v_S \simeq v_S^*$ by linear combinations of the degrees of freedom (using interpolation).
3. Build a linear equation for the degrees of freedom

$$(u_E(\boldsymbol{\theta}) - u_W(\boldsymbol{\theta}))\, h_y + (v_N(\boldsymbol{\theta}) - v_S(\boldsymbol{\theta}))\, h_x = 0\ . \tag{5.8}$$

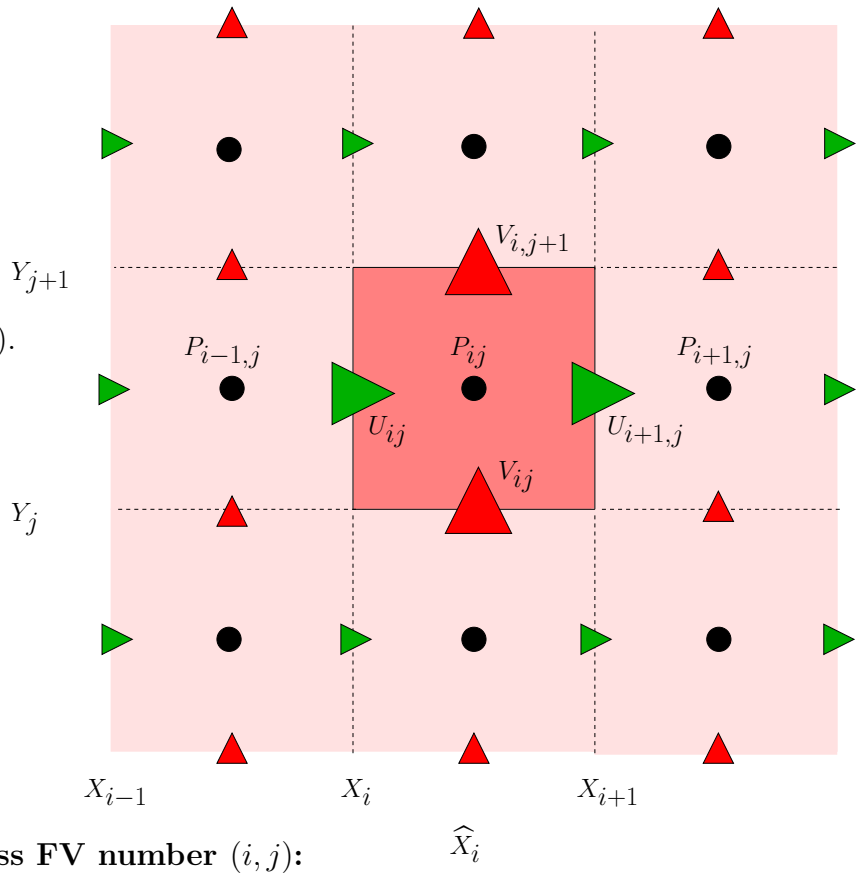
Caution: The basic methodology above **can fail**. In what sense? **Depending on the choice of $\boldsymbol{\theta}$, the equations corresponding to the different finite volumes may become linearly dependent!** This gives rise to **spurious pressure modes**, which polute the pressure field.

$$\begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}p} \\ A_{p\mathbf{u}} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}_{\mathbf{u}} \\ \boldsymbol{\theta}_p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix} \qquad \begin{matrix} \text{momentum} \\ \text{incompressibility} \end{matrix} \tag{5.9}$$

Example: If the degrees of freedom of $\mathbf{u}$ are taken at the centers of the cells (collocated grid), then the resulting matrix is rank-defficient.

MAC discretization:

- $\theta_u \to u$ at vertical faces.
- $\theta_v \to v$ at horizontal faces.
- Darker pink is **mass FV** numbered as (i, j).
- The **intervening unknowns** are **larger**.



Discrete incompressibility equation for mass FV number (i, j):

$$(U_{i+1,j} - U_{ij})\,(Y_{j+1} - Y_j) \;+\; (V_{i,j+1} - V_{i,j})\,(X_{i+1} - X_i) \;=\; 0 \qquad (5.10)$$

Discrete incompressibility equation for mass FV number (i, j):

$$(U_{i+1,j} - U_{ij})\,(Y_{j+1} - Y_j) \;+\; (V_{i,j+1} - V_{i,j})\,(X_{i+1} - X_i) \;=\; 0 \tag{5.11}$$

- The finite volumes for this equation are centered at pressure nodes and have as unknowns u_E, u_W, v_N and v_S, exactly as needed (for second order) and thus **requiring no interpolation**.
- Dividing by $h_x h_y \ldots$

$$\frac{U_{i+1,j} - U_{ij}}{X_{i+1} - X_i} \;+\; \frac{V_{i,j+1} - V_{i,j}}{Y_{j+1} - Y_j} \;=\; 0 \tag{5.12}$$

  one puts in evidence the link with $\nabla \cdot \mathbf{u} = 0$.
- In matrix form (ignoring boundary conditions and with uniform mesh),

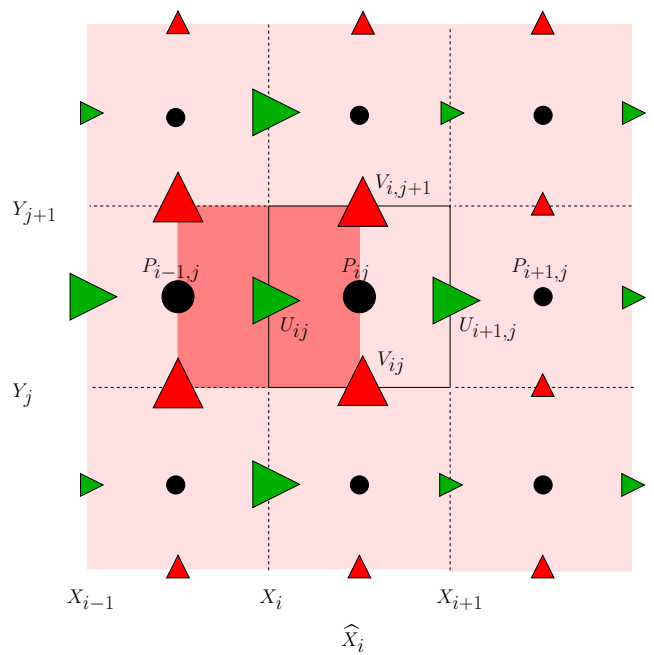
$$\underline{\underline{D_x}}\;\underline{u} \;+\; \underline{\underline{D_y}}\;\underline{v} \;=\; 0\;. \tag{5.13}$$

5.3 Momentum along the x_1 direction

The x-momentum equation reads

$$\int_V \rho\,\partial_t u\; dV + \int_{\partial V} \left(\rho u\,\mathbf{u}\cdot\check{\mathbf{n}} + p\,n_x - 2\,\mu\,\partial_x u\,n_x - \mu\,\partial_x v\,n_y - \mu\,\partial_y u\,n_y\right)\; dS \;=\; \int_V f_x\; dV \tag{5.14}$$

Now we particularize for a rectangle of sides h_x and h_y.



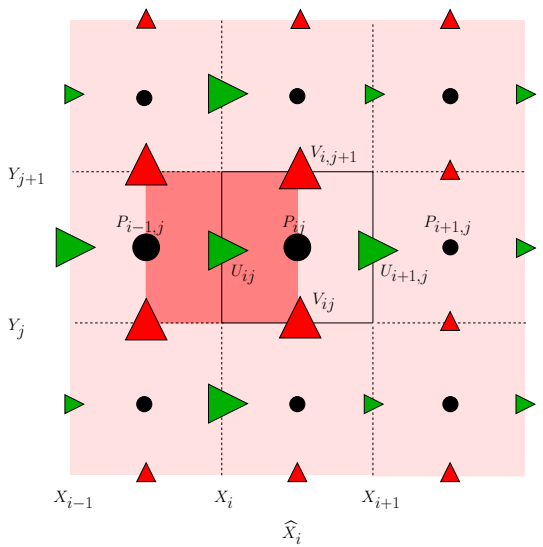
For the pressure term,

$$\int_E p\, n_x \, dS \;=\; p_E \, h_y \;, \tag{5.15}$$

$$\int_W p\, n_x \, dS \;=\; -p_W \, h_y \;, \tag{5.16}$$

$$\int_N p\, n_x \, dS \;=\; 0 \;, \tag{5.17}$$

$$\int_S p\, n_x \, dS \;=\; 0 \;. \tag{5.18}$$



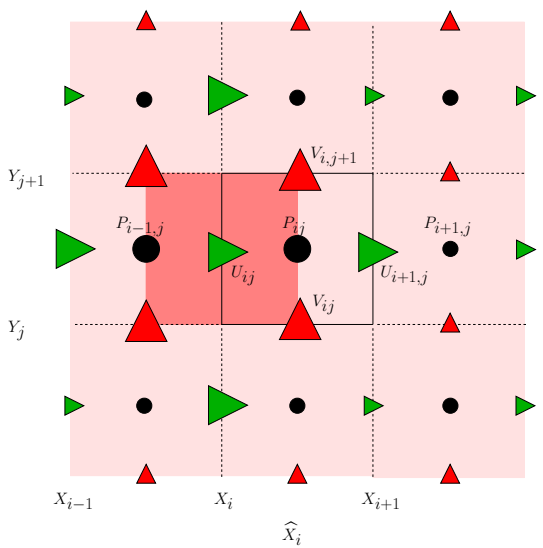
For the viscous term, integrating edge by edge,

$$-\left(\int_E \mu\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\check{\mathbf{n}}dS\right)_x = -\int_E 2\mu\,\partial_x u\;dy = -2\,\mu\,\partial_x u|_E\,h_y \;=\; \text{(VE)}\;, \tag{5.19}$$

$$-\left(\int_W \mu\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\check{\mathbf{n}}dS\right)_x = +\int_W 2\mu\,\partial_x u\;dy = 2\,\mu\,\partial_x u|_W\,h_y \;=\; \text{(VW)}\;, \tag{5.20}$$

$$-\left(\int_N \mu\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\check{\mathbf{n}}dS\right)_x = -\int_N \mu\,(\partial_y u+\partial_x v)\;dx = -\,\mu\,(\partial_y u+\partial_x v)|_N\,h_x \;=\; \text{(VN)}\;, \tag{5.21}$$

$$-\left(\int_S \mu\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\check{\mathbf{n}}dS\right)_x = +\int_S \mu\,(\partial_y u+\partial_x v)\;dx = \;\mu\,(\partial_y u+\partial_x v)|_N\,h_x \;=\; \text{(VS)}\;. \tag{5.22}$$



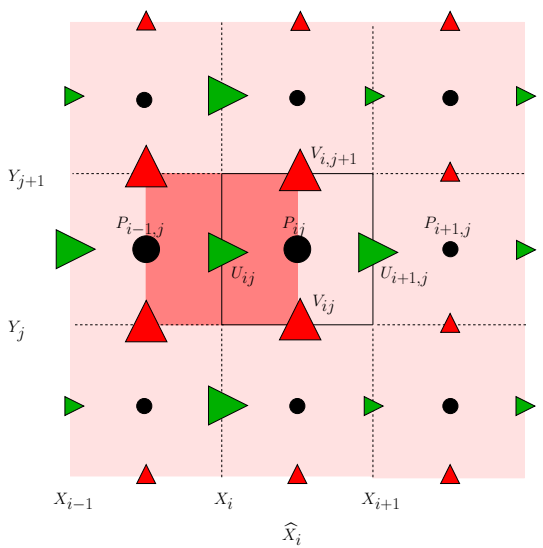
For the inertia term, the x component being $\int_{\partial V} \rho\, u\, \mathbf{u}\cdot\check{\mathbf{n}}\; dS$,

$$\int_E \rho\, u\; \mathbf{u}\cdot\check{\mathbf{n}}\; dS \;=\; \rho\, u_E^2\, h_y\;, \tag{5.23}$$

$$\int_W \rho\, u\; \mathbf{u}\cdot\check{\mathbf{n}}\; dS \;=\; -\rho\, u_W^2\, h_y\;, \tag{5.24}$$

$$\int_N \rho\, u\; \mathbf{u}\cdot\check{\mathbf{n}}\; dS \;=\; \rho\, u_N\, v_N\, h_x\;, \tag{5.25}$$

$$\int_S \rho\, u\; \mathbf{u}\cdot\check{\mathbf{n}}\; dS \;=\; -\rho\, u_S\, v_S\, h_x\;. \tag{5.26}$$

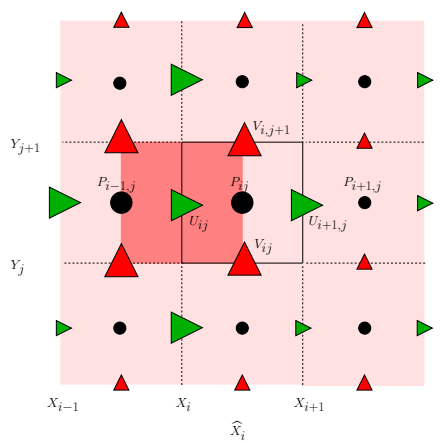


Now we consider the staggered arrangement of degrees of freedom, in which the unknowns are

- For pressure: $p_E = P_{ij}$, $p_W = P_{i-1,j}$.
- For u: $u_P = U_{ij}$, $u_{EE} = U_{i+1,j}$, $u_{WW} = U_{i-1,j}$, $u_{NN} = U_{i,j+1}$, $u_{SS} = U_{i,j-1}$.
- For v: $v_{NE} = V_{i,j+1}$, $v_{NW} = V_{i-1,j+1}$, $v_{SE} = V_{ij}$, $v_{SW} = V_{i-1,j}$.

and interpolate to approximate each term.
Then, the **pressure term** becomes

$$(\mathrm{P}) \;=\; \int_{\partial V} p\, n_x \, dS \simeq (p_E - p_W)\, h_y \;=\; (P_{ij} - P_{i-1,j})\,(Y_{j+1} - Y_j)\;. \tag{5.27}$$



In matrix form (uniform mesh)

$$(\mathrm{P}) \;=\; h_x\, h_y\, \underline{\underline{G_x}}\; \underline{p}\;. \tag{5.28}$$

The **viscous term** becomes (notice that μ may vary from face to face)

- East face:

$$(\mathrm{VE}) \simeq -2\,\mu_E\,\partial_x u|_E\,h_y \;\simeq\; -2\,\mu_E\,\frac{u_{EE}-u_P}{x_{EE}-x_P}\,h_y = -2\,\mu_E\,\frac{U_{i+1,j}-U_{ij}}{X_{i+1}-X_i}\,(Y_{j+1}-Y_j) \tag{5.29}$$

- West face:

$$(\mathrm{VW}) \simeq 2\,\mu_W\,\partial_x u|_W\,h_y \;\simeq\; 2\,\mu_W\,\frac{u_{WW}-u_P}{x_{WW}-x_P}\,h_y = 2\,\mu_W\,\frac{U_{ij}-U_{i-1,j}}{X_i-X_{i-1}}\,(Y_{j+1}-Y_j) \tag{5.30}$$

- North face (**beware of hat coordinates!**):

$$(\mathrm{VN}) \simeq -\,\mu_N\,(\partial_y u+\partial_x v)|_N\,h_x \;\simeq\; -\,\mu_N\left(\frac{U_{i,j+1}-U_{ij}}{\widehat{Y}_{j+1}-\widehat{Y}_j}+\frac{V_{i,j+1}-V_{i-1,j+1}}{\widehat{X}_i-\widehat{X}_{i-1}}\right)(\widehat{X}_i-\widehat{X}_{i-1}) \tag{5.31}$$

- South face:

$$(\mathrm{VS}) \simeq \mu_S\,(\partial_y u+\partial_x v)|_N\,h_x \;\simeq\; \mu_S\left(\frac{U_{i,j}-U_{i,j-1}}{\widehat{Y}_j-\widehat{Y}_{j-1}}+\frac{V_{i,j}-V_{i-1,j}}{\widehat{X}_i-\widehat{X}_{i-1}}\right)(\widehat{X}_i-\widehat{X}_{i-1}) \tag{5.32}$$

Notice that **all faces contribute positively to the diagonal**.

The **inertia term**, integrated over the cell boundary, reads

$$\text{(IX)} \;=\; \rho\left(u_E^2 h_y - u_W^2 h_y + u_N v_N h_x - u_S v_S h_x\right) \;, \tag{5.33}$$

which we complement with a centered interpolation:

$$u_E = \frac{u_{EE}+u_P}{2} = \frac{U_{i+1,j}+U_{ij}}{2} \;, \qquad u_W = \frac{u_{WW}+u_P}{2} = \frac{U_{i-1,j}+U_{ij}}{2} \;, \tag{5.34}$$

$$u_N = \frac{u_{NN}+u_P}{2} = \frac{U_{i,j+1}+U_{ij}}{2} \;, \qquad u_S = \frac{u_{SS}+u_P}{2} = \frac{U_{i,j-1}+U_{ij}}{2} \;, \tag{5.35}$$

$$v_N = \frac{v_{NE}+v_{NW}}{2} = \frac{V_{i,j+1}+V_{i-1,j+1}}{2} \;, \qquad v_S = \frac{v_{SE}+v_{SW}}{2} = \frac{V_{i,j}+V_{i-1,j}}{2} \;. \tag{5.36}$$

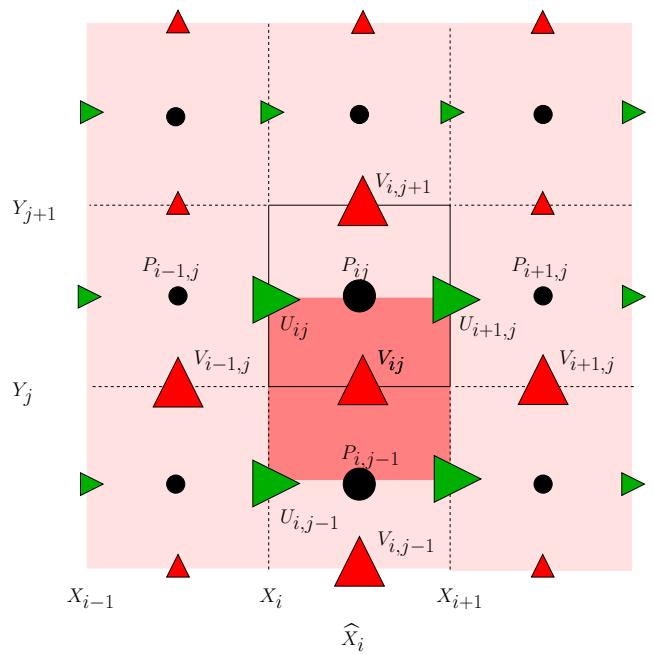
Notice that the vector of values of u_E at all east faces of finite volumes centered in nodes of u can be built as $\underline{u_E} = \underline{\underline{A^E_{xu}}}\,\underline{u}$, where $\underline{\underline{A^E_{xu}}}$ is an interpolation matrix. Similar matrix operations can be devised for the other necessary quantities.

5.4 Momentum along the x_2 direction

The y-momentum equation reads

$$\int_V \rho\, \partial_t v \; dV + \int_{\partial V} \left(\rho v\, \mathbf{u}\cdot\check{\mathbf{n}} + p\, n_y - 2\,\mu\, \partial_y v\, n_y - \mu\, \partial_x v\, n_x - \mu\, \partial_y u\, n_x\right) \; dS \;\; = \;\; \int_V f_y \; dV \tag{5.37}$$

Exo. 5.1 *Deduce the MAC discretization of the pressure and viscous terms of the x_2-momentum equation, with the finite volume depicted in the Figure.*



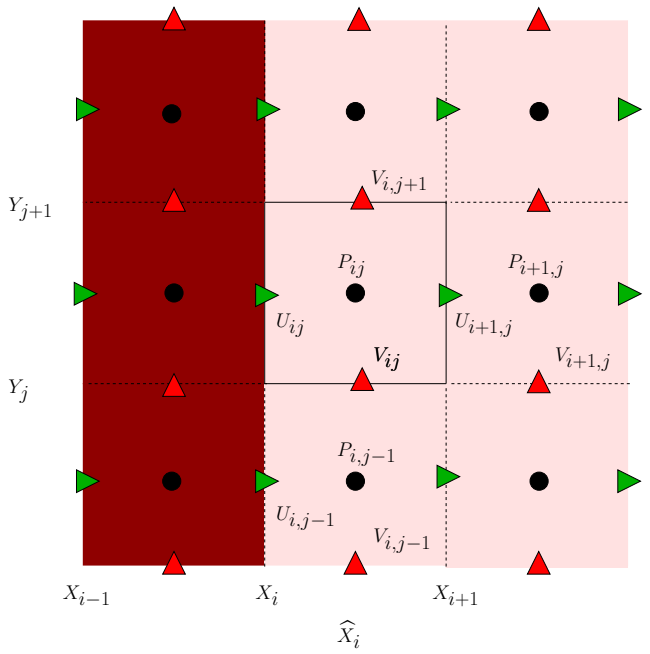
5.5 Boundary conditions and a software for Stokes flow

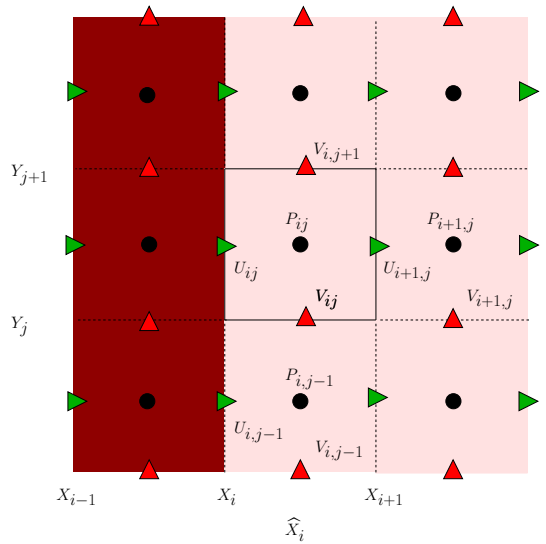
In file `codemac.m` (which also uses `ij2ng.m`) you will find an Octave implementation of the MAC method for **Stokes flow** (in which inertia terms have been neglected).
Some characteristics:

- The encompassing domain $\Omega_0 = (0, L_x) \times (0, L_y)$ is discretized with $(N_x = N_1 + 2) \times (N_y = N_2 + 2)$ "pressure" cells. The first and last rows and columns are dummy cells that lie outside Ω_0 and are only used to specify the boundary conditions.
- The first `dimU` $(= (N_x + 1)N_y)$ equations correspond to conservation of x_1-momentum, and correspondingly the first `dimU` unknowns are the u-values at vertical faces.
- The next `dimV` $(= N_x(N_y + 1))$ equations correspond to conservation of x_2-momentum, and correspondingly the next `dimV` unknowns are the v-values at horizontal faces.
- The last `dimP` $(= N_x N_y)$ equations correspond to conservation of mass (incompressibility). The last `dimP` unknowns are the p-values at cell centers.
- A **mask**, `mask(1:Nx,1:Ny)` defines whether a cell is fluid or boundary. **This implies that all boundaries coincide with pressure-cell boundaries**.

Imposition of Dirichlet boundary conditions

Consider the boundary shown in the picture. **In brown is a wall where both components of the velocity are imposed**. How will the momentum and mass equations be affected?



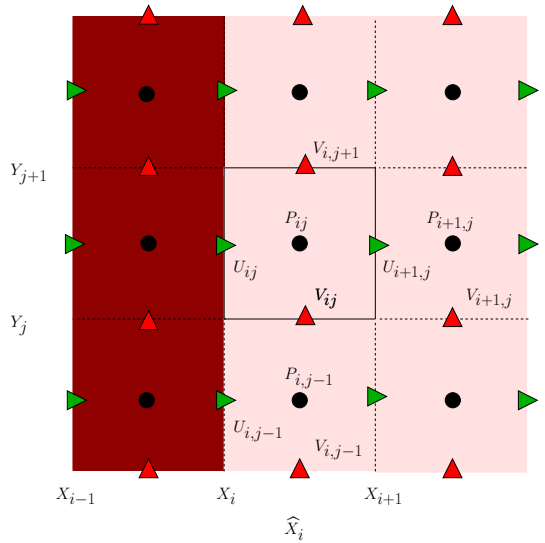


- The unknowns that lie entirely in the brown region become **dummy** variables. They must be assigned some arbitrary value.
- The **mass equations** will not be affected at all.
- The x**-momentum equations** will be only slightly affected: Just specify the value of U_{ij} to the imposed value.
- The y**-momentum equations** need more consideration. Specifically, for FV number ij in the Figure,

$$\int_W \mu\, \partial_x v \; dy \;\; \simeq \;\; \mu_W \, (Y_{j+1} - Y_j) \, \frac{V_{ij} - V_W}{(X_{i+1} - X_i)/2}$$

where V_W is the imposed value at the point (X_i, Y_j).

Imposition of normal forces



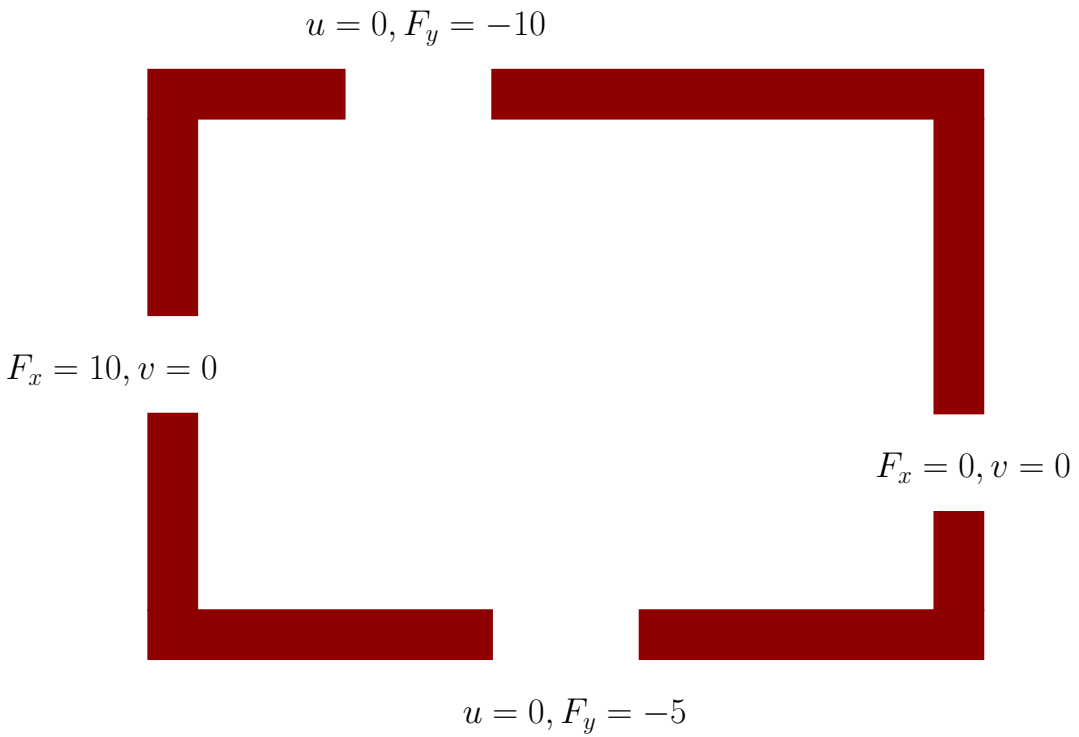
If the wall imposes tangential (vertical) velocity as before, but normal (horizontal) **force** instead, **then the x-momentum equation changes significantly**:

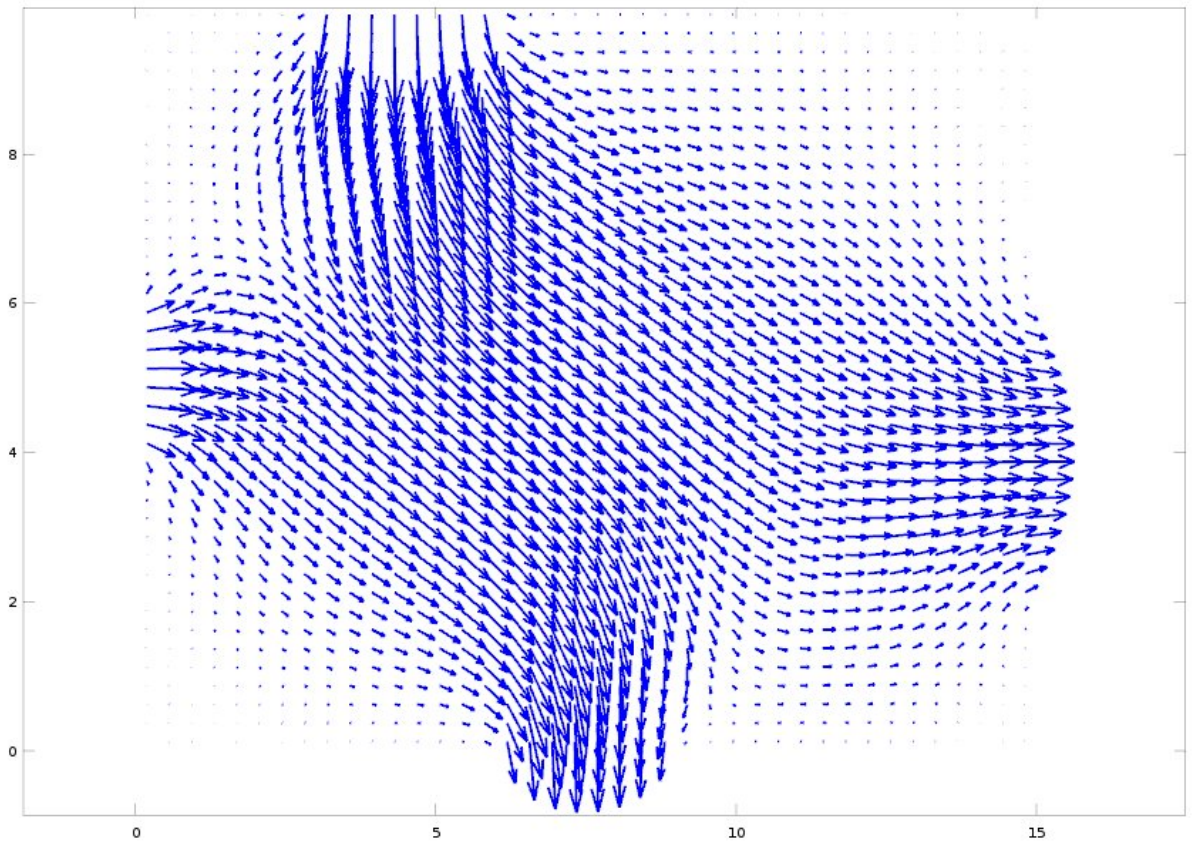
- The x-momentum finite volume is reduced by half: $(X_i, \widehat{X}_i) \times (Y_j, Y_{j+1})$ instead of $(\widehat{X}_{i-1}, \widehat{X}_i) \times (Y_j, Y_{j+1})$.
- The integral on the West face becomes $-F_W\,(Y_{j+1} - Y_j)$, which moves to the right-hand side, where F_W is the normal force imposed.
- The North and South integrals of $\mu\,\partial_x v$ require values of v at (X_i, Y_{j+1}) and (X_i, Y_j), which are taken from the imposed tangential velocity.

Exo. 5.2 *Write down the complete x-momentum discrete equation (Stokes flow) for volume number (i, j), assuming v_W and F_W given.*

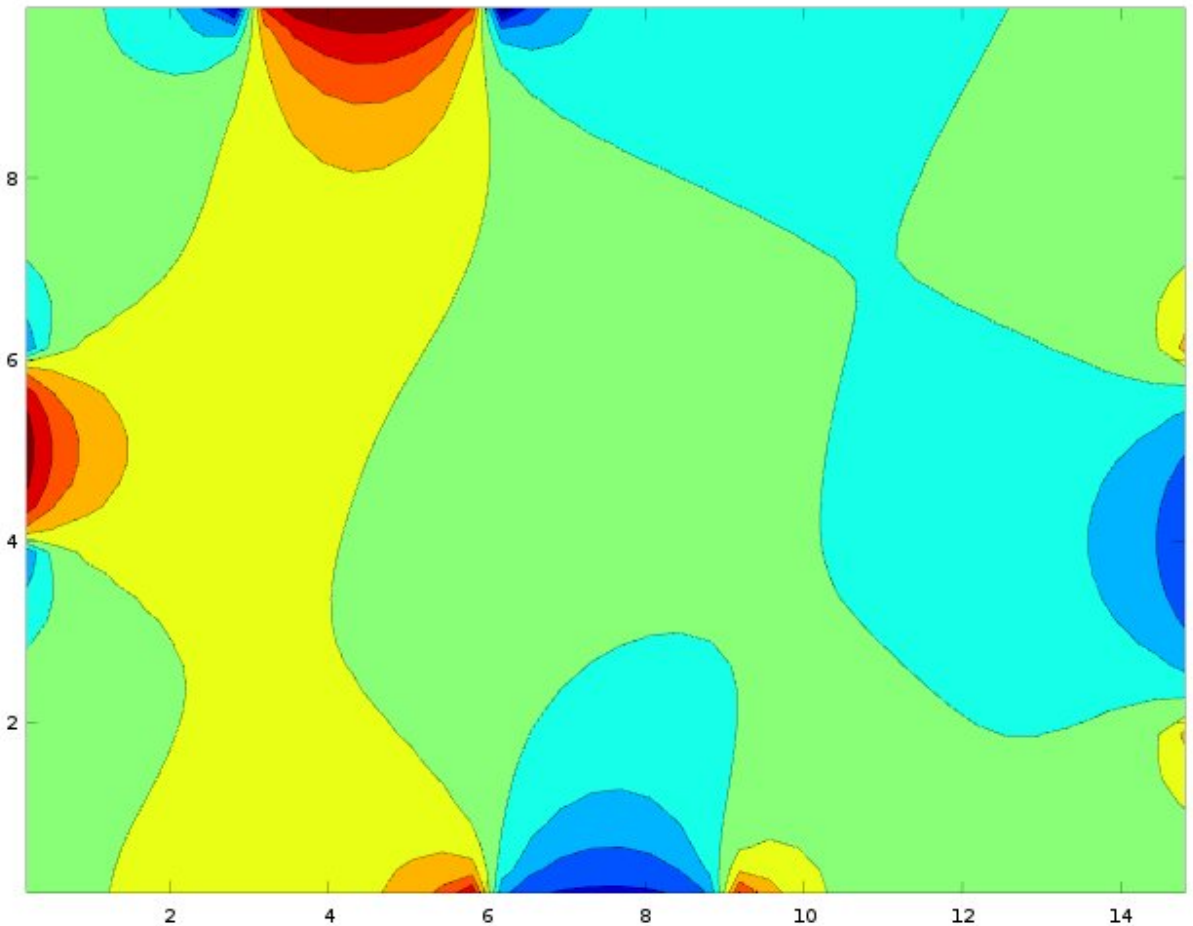
```
nguP=ij2ng(Nx+1,i,j); mskl=mask(i-1,j); mskr=mask(i,j); ## masks of left and right half cells
.......
elseif (mskl!=0) ## left-half of cell is boundary, imposes x force and y velocity
    ### W boundary (rhs+=integral of force)
    rhs(nguP)=rhs(nguP)+dy(j)*vbcx(mskl);
    ### E boundary (integral of p-2*muE*du/dx)
    muE=mu; nguEE=ij2ng(Nx+1,i+1,j); ngpE=ij2ng(Nx,i,j)+dimU+dimV;
    Ag(nguP,nguP)=Ag(nguP,nguP)+2*muE*dy(j)/dx(i);
    Ag(nguP,nguEE)=Ag(nguP,nguEE)-2*muE*dy(j)/dx(i);
    Ag(nguP,ngpE)=dy(j);
    ### N boundary (integral of -mu(dv/dx), E half of upper cell bound.)
    muN=mu; vN=0.5*(vbcy(mskl)+vbcy(mask(i-1,j+1)));
    nguNN=ij2ng(Nx+1,i,j+1);ngvNE=ij2ng(Nx,i,j+1)+dimU;
    Ag(nguP,ngvNE)=Ag(nguP,ngvNE)-muN*(dx(i)/2)/(dx(i)/2);
    rhs(nguP)=rhs(nguP)-muN*(dx(i)/2)/(dx(i)/2)*vN;
    ### N boundary (integral of -mu(du/dy), E half)
    Ag(nguP,nguP) =Ag(nguP,nguP) +muN*0.5*dx(i)/dyh(j);
    Ag(nguP,nguNN)=Ag(nguP,nguNN)-muN*0.5*dx(i)/dyh(j);
    ### S boundary (integral of mu(dv/dx), E half)
    muS=mu; vS=0.5*(vbcy(mskl)+vbcy(mask(i-1,j-1)));
    nguSS=ij2ng(Nx+1,i,j-1);ngvSE=ij2ng(Nx,i,j)+dimU;
    Ag(nguP,ngvSE)=Ag(nguP,ngvSE)+muS*(dx(i)/2)/(dx(i)/2);
    rhs(nguP)=rhs(nguP)+muS*(dx(i)/2)/(dx(i)/2)*vS;
    ### S boundary (integral of mu(du/dy), E half)
    Ag(nguP,nguP) =Ag(nguP,nguP) +muS*0.5*dx(i)/dyh(j-1);
    Ag(nguP,nguSS)=Ag(nguP,nguSS)-muS*0.5*dx(i)/dyh(j-1);
```

An example:





Velocity field



Pressure field

5.6 Laplacian form of the viscous term

The general viscous term $-\nabla \cdot \left[\mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)\right]$ can, when the viscosity is constant, be replaced by the simpler term $-\mu \nabla^2 \mathbf{u}$.

Proof: $\nabla \cdot \nabla \mathbf{u}^T = u_{j,ij} \mathbf{e}_i = \nabla \left(\nabla \cdot \mathbf{u}\right) = 0$.

The MAC discretization allows for this simplification to be carried out at the discrete level. Consider for simplicity the uniform-spacing case. Then

$$\left(U_{i+1,j} - U_{ij}\right) h_y \; + \; \left(V_{i,j+1} - V_{ij}\right) h_x = 0 \; , \text{ and } \qquad \left(U_{ij} - U_{i-1,j}\right) h_y \; + \; \left(V_{i-1,j+1} - V_{i-1,j}\right) h_x = 0 \; . \quad (5.38)$$

Exo. 5.3 *Prove that the viscous contribution ends up being simplified to*

$$(VX) \;=\; -\mu \left(\frac{U_{i+1,j} - U_{ij}}{h_x} h_y + \frac{U_{i-1,j} - U_{ij}}{h_x} h_y + \frac{U_{i,j+1} - U_{ij}}{h_y} h_x + \frac{U_{i,j-1} - U_{ij}}{h_y} h_x \right) \qquad (5.39)$$

and verify that this corresponds to a finite volume approximation of $-\mu \nabla^2 u$.

In matrix form, $\underline{\underline{L}}$ being the Laplacian matrix,

$$(\mathrm{VX}) \;=\; -\mu \, h_x \, h_y \, \underline{\underline{L}} \, \underline{u} \; . \qquad (5.40)$$

5.7 Full set of equations in matrix form

To visualize the full set of equations, let us consider the uniform spacing case (after dividing by $h_x h_y$) with the Laplacian form of the viscous term:

$$\rho \frac{d}{dt} \underline{u} + \underline{\underline{G_x}}\, \underline{p} - \mu \underline{\underline{L}}\, \underline{u} + \underline{\mathrm{IX}}(\underline{u}, \underline{v}) = \underline{f_x} \tag{5.41}$$

$$\rho \frac{d}{dt} \underline{v} + \underline{\underline{G_y}}\, \underline{p} - \mu \underline{\underline{L}}\, \underline{v} + \underline{\mathrm{IY}}(\underline{u}, \underline{v}) = \underline{f_y} \tag{5.42}$$

$$\underline{\underline{D_x}}\, \underline{u} + \underline{\underline{D_y}}\, \underline{v} = 0 \tag{5.43}$$

which can even be simplified to, with some additional quite natural notations,

$$\rho \frac{d}{dt} \underline{U} + \underline{\underline{G}}\, \underline{p} - \mu \underline{\underline{L}}\, \underline{U} + \underline{\mathrm{I}}(\underline{U}) = \underline{f} \tag{5.44}$$

$$\underline{\underline{D}}\, \underline{U} = 0 \tag{5.45}$$

This is a so-called *differential-algebraic equation* (DAE) system.

5.8 Monolithic system

The monolithic approach solves for $\underline{U}^{n+1}$ and $\underline{p}^{n+\theta}$ in a straightforward manner:

$$\rho \frac{\underline{U}^{n+1} - \underline{U}^{n}}{\Delta t} + \underline{\underline{G}}\, \underline{p}^{n+\theta} - \mu\, \underline{\underline{L}}\, \underline{U}^{n+\theta} + \underline{\mathrm{I}}(\underline{U}^{n+\theta}) = \underline{f}^{n+\theta} \tag{5.46}$$

$$\underline{\underline{D}}\, \underline{U}^{n+1} = 0 \tag{5.47}$$

- Matrices $\underline{\underline{G}}$, $\underline{\underline{D}}$ and $\underline{\underline{L}}$ are built only once.
- The term $\underline{I}(\underline{U}^{n+\theta})$ is **nonlinear**. Linearize by Newton, fixed-point, etc.
- This approach is efficient for **Stokes flow** mainly.
- It is considered expensive because all unknowns are solved at once (big matrices).

5.9 Projection method

Chorin's (1968) original projection method was defined as follows:

- **Momentum predictor**:

$$\rho \frac{\hat{\underline{U}}^{n+1} - \underline{U}^n}{\Delta t} - \mu \underline{\underline{L}}\, \underline{U}^n + \underline{\mathrm{I}}(\underline{U}^n) = \underline{f}^n \tag{5.48}$$

  The resulting $\hat{\underline{U}}^{n+1}$ does not satisfy $\underline{\underline{D}}\, \hat{\underline{U}}^{n+1} = 0$, it has to be *projected back* on the discretely-incompressible space.

- **Pressure Poisson equation**:

$$\underline{\underline{D}}\, \underline{\underline{G}}\, \underline{p}^{n+1} = \frac{\rho}{\Delta t}\, \underline{\underline{D}}\, \hat{\underline{U}}^{n+1} \tag{5.49}$$

  The product $\underline{\underline{D}}\, \underline{\underline{G}}$ is a matrix that, leaving aside boundary conditions, coincides with $\underline{\underline{L}}$ (the discrete Laplacian matrix). This is a salient property of the MAC discretization.

- **Velocity correction**:

$$\underline{U}^{n+1} = \hat{\underline{U}}^{n+1} - \frac{\Delta t}{\rho}\, \underline{\underline{G}}\, \underline{p}^{n+1} \tag{5.50}$$

- Eliminating $\underline{\hat{U}}^{n+1}$ one gets

$$\rho \frac{\underline{U}^{n+1} - \underline{U}^n}{\Delta t} + \underline{\underline{G}}\, \underline{p}^{n+1} - \mu\, \underline{\underline{L}}\, \underline{U}^n + \underline{\mathrm{I}}(\underline{U}^n) = \underline{f}^n \tag{5.51}$$

$$\underline{\underline{D}}\, \underline{U}^{n+1} = 0 \tag{5.52}$$

  which shows that the algorithm is consistent, but clearly of first order in Δt because the viscous and inertia term are evaluated at t_n.

- Eliminating $\underline{U}^n$ instead one gets

$$\rho \frac{\underline{\hat{U}}^{n+1} - \underline{\hat{U}}^n}{\Delta t} + \underline{\underline{G}}\, \underline{p}^n - \mu\, \underline{\underline{L}}\, \underline{U}^n + \underline{\mathrm{I}}(\underline{U}^n) = \underline{f}^n \tag{5.53}$$

$$\underline{\underline{D}}\, \underline{\hat{U}}^{n+1} - \frac{\Delta t}{\rho}\, \underline{\underline{D}}\, \underline{\underline{G}}\, \underline{p}^{n+1} = 0 \tag{5.54}$$

  which shows the appearance of a **pressure Laplacian** in the incompressibility equation, with coefficient $\Delta t/\rho$.

- In collocated grids, $\underline{\underline{DG}} \neq \underline{\underline{L}}$. However, if nevertheless the pressure is computed from

$$\underline{\underline{L}}\, \underline{p}^{n+1} = \frac{\rho}{\Delta t}\, \underline{\underline{D}}\, \underline{\hat{U}}^{n+1} \tag{5.55}$$

  there is a stabilization effect on the incompressibility constraint, so that **projection schemes work well with collocated grids**, if $\Delta t/\rho$ is not too small.

5.10 ABCN time discretization

The following combination of Adams-Bashforth scheme for time discretization of the inertia terms, combined with Crank-Nicolson scheme for the viscous term, is a popular method with formally second order accuracy in time.

- **Momentum predictor**:

$$\rho \frac{\underline{\hat{U}}^{n+1} - \underline{U}^n}{\Delta t} - \frac{\mu}{2} \underline{\underline{L}} \, (\underline{\hat{U}}^{n+1} + \underline{U}^n) + \frac{3}{2} \, \underline{\mathrm{I}}(\underline{U}^n) - \frac{1}{2} \, \underline{\mathrm{I}}(\underline{U}^{n-1}) = \underline{f}^{n+\frac{1}{2}} \tag{5.56}$$

- **Pressure Poisson equation**:

$$\underline{\underline{L}} \, \underline{\phi}^{n+1} = \frac{\rho}{\Delta t} \, \underline{\underline{D}} \, \underline{\hat{U}}^{n+1} \tag{5.57}$$

- **Velocity correction**:

$$\underline{U}^{n+1} = \underline{\hat{U}}^{n+1} \; - \; \frac{\Delta t}{\rho} \, \underline{\underline{G}} \, \underline{\phi}^{n+1} \tag{5.58}$$

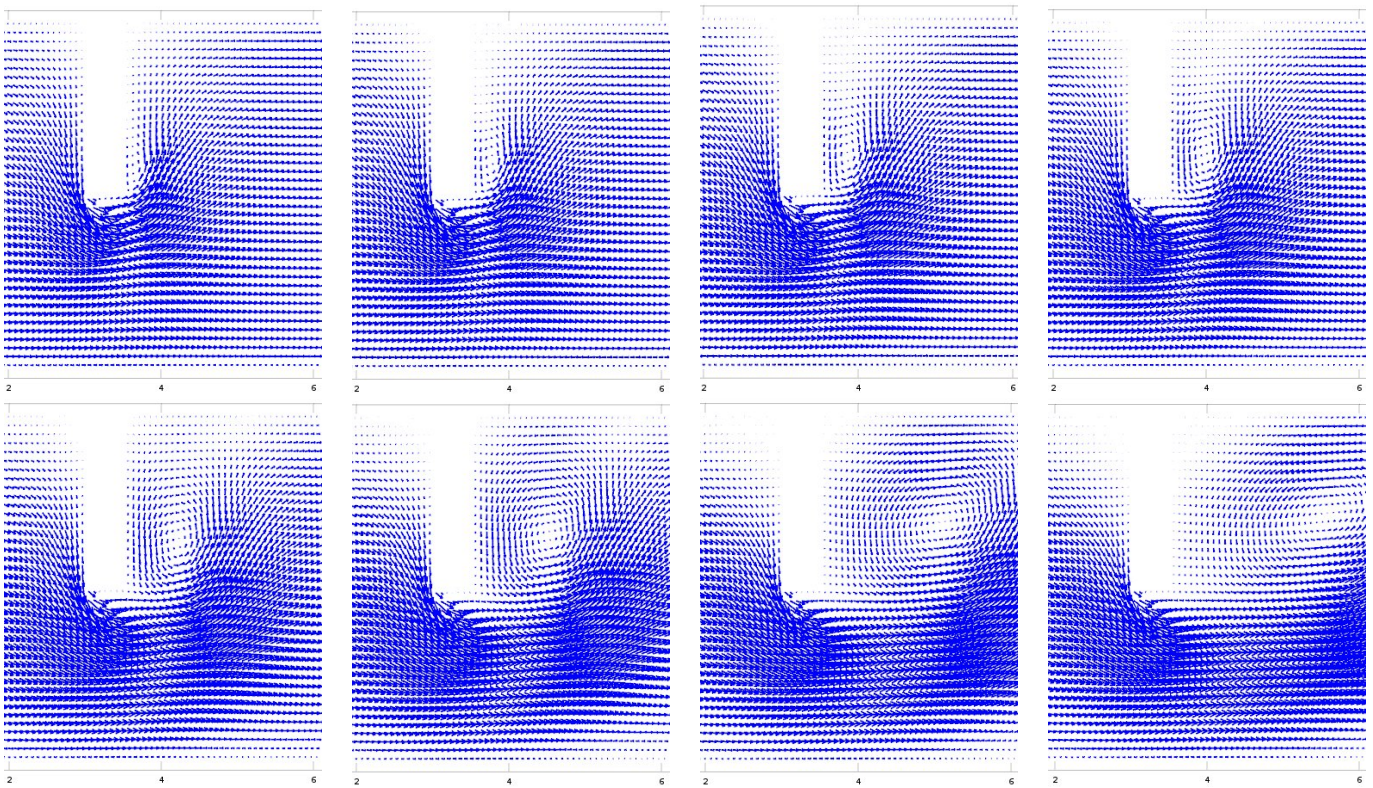
Exo. 5.4 *Read further details from Chapter 2 of Prosperetti & Tryggvasson.*

Exo. 5.5 *Read Chapter 6 of Wesseling. Explain the necessity of pressure stabilization in the case of collocated discretization (all variables located at cell centers). What is the key modification that stabilizes the scheme?*

Exo. 5.6 *Read Sousa et al (2015) for more projection-like methods and a warning about the usage of projection methods for low inertia flows.*

Code NSMAC

You will find in the site a vector implementation of the MAC-ABCN discretization, for the problem of flow past a partial obstruction.



Development of the velocity field

Home reading:

For next week, reed Lectures 8 and 10 of Bakker. Complement with Chapter 9 of Ferziger & Peric and with the material available at the site.

Discuss the following topics:

1. What is turbulence?
2. What happens to flows as the Reynolds number is increased?
3. Transition to turbulence.
4. Objectives of turbulence modeling.
5. Direct Numerical Simulation.
6. Velocity decomposition. Averaging. Filtering.
7. Large Eddy Simulation.
8. Reynolds Averaged Navier-Stokes simulation.
9. Reynolds stresses.
10. Boussinesq hypothesis.
11. Turbulent viscosity.
12. The $k - \varepsilon$ model.

6 A bit of turbulence

6.1 Flow in a long pipe

A long pipe conveys water between two reservoirs that are far apart. The inclination of the pipe is s (in meters of descent per meter of length) and its diameter D. Compute the velocity field in the pipe and the flow rate.

Applying the general expression for conservation of momentum in fully developed flow (3.7) to the case in which ω is the circle of radius r and the flow steady we obtain

$$0 = -\mathcal{G}\,\pi\,r^2 + r\int_0^{2\pi}\tau(r,\theta)\;d\theta \tag{6.1}$$

where τ is the radial shear stress along x_3, given by $\boldsymbol{\tau}\cdot\check{\mathbf{e}}_r$. Because of the symmetry, τ does not depend on θ, which gives

$$\tau(r)\,2\pi\,r = \mathcal{G}\,\pi\,r^2\ . \tag{6.2}$$

where σ_{rz} is the shear stress along z (the axial direction) on surfaces with normal $\check{\boldsymbol{\nu}} = \check{\mathbf{r}}$. The inclination generates the pressure gradient

$$\mathcal{G} = -s\,\rho\,g\ , \tag{6.3}$$

and from the Newtonian law

$$\tau(r) = \mu\,\frac{dw}{dr}\ . \tag{6.4}$$

Notice that

$$\frac{1}{r}\,\frac{d}{dr}\left(\mu\, r\,\frac{dw}{dr}\right) = \mathcal{G} \tag{6.5}$$

in agreement with (3.14), since the left-hand side of (6.5) is $\nabla\cdot(\mu\,\nabla w)$ in cylindrical coordinates and we have assumed steady flow. This is the equation that determines w, with boundary condition $w(R) = 0$. The condition $w'(0) = 0$ is generally also imposed, but truly speaking $r = 0$ is not a boundary.

Anyway, the governing equation is

$$\frac{dw}{dr}(r) = -\,\frac{s\,\rho\,g}{2}\,\frac{r}{\mu(r)} \tag{6.6}$$

which can be integrated with initial condition $w(r=0) = w_{\max}$ to yield

$$w(r) = w_{\max}\;-\;\frac{s\,\rho\,g}{2}\int_0^r \frac{r'dr'}{\mu(r')}\;. \tag{6.7}$$

The unknown $w_{\max}$ can be computed from $w(R) = 0$, namely

$$w_{\max}\;=\;\frac{s\,\rho\,g}{2}\int_0^R \frac{r'dr'}{\mu(r')}\;. \tag{6.8}$$

If the viscosity is constant one recovers the familiar parabolic Poiseuille profile

$$w(r) = w_{\max} - \;\frac{s\,\rho\,g}{4\,\mu}\;r^2 \tag{6.9}$$

with

$$w_{\max} = \frac{s\,\rho\,g}{16\,\mu}\;D^2\;. \tag{6.10}$$

Assuming $D = 0.5$ m and a gentle slope of $s = 10^{-2}$ (ten meters per kilometer), since $\mu = 10^{-3}$ Pa-s and $\rho = 1000$ kg/m^3 one gets

$$w_{\max} = 1530 \text{ m/s} \;=\; 5512 \text{ km/h !!!!!} \tag{6.11}$$

6.2 Turbulence

It is obvious that the huge velocity obtained above does not occur in reality. The parallel flow is indeed a solution of the conservation equations, but it is an *unstable* solution. Both mathematically and physically one observes a flow that is neither steady nor parallel, that is called *turbulent*.

- Turbulent flows are stochastic. They are described with the tools of statistical theory. Though the **instantaneous** values of velocity and pressure are randomic, the **mean** values of the variables are quite deterministic.
- These mean values (of velocity, of pressure, of force on solid surfaces, etc.) are in fact what engineers are most interested in. If the boundary conditions do not depend on time, the mean values also do not depend on time, as would be the case, in our pipe example, some seconds after the valve connecting the two reservoirs is opened.
- It is customary to decompose all variables into mean and fluctuating components, e.g.,

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \qquad p = \bar{p} + p' \quad . \tag{6.12}$$

Inserting this into the momentum balance equation and taking the mean, one arrives at

$$0 = \int_V \bar{\mathbf{f}} \; dV \; + \; \int_{\partial V} (-\bar{p} + \mu \, (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T) - \rho \, \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \rho \, \overline{\mathbf{u}' \otimes \mathbf{u}'}) \cdot \check{\mathbf{n}} \; dS \; . \tag{6.13}$$

Exo. 6.1 *Verify the previous assertion.*

We observe that *the averaged equation is the same as the original equation if the so-called Reynolds stress tensor is added to the average stresses:*

$$\boldsymbol{\sigma} \longleftarrow \boldsymbol{\sigma}(\nabla \bar{\mathbf{u}}, \bar{p}) + \boldsymbol{\sigma}^{\mathrm{Re}} \; , \qquad \text{with} \qquad \boldsymbol{\sigma}^{\mathrm{Re}} = -\rho \, \overline{\mathbf{u}' \otimes \mathbf{u}'} \; . \tag{6.14}$$

- Equation (6.2) then becomes, for a long circular pipe,

$$\left[\mu\,\partial_r\bar{w} - \rho\,\overline{u_r' u_z'}\right]\,2\pi\,r = \mathcal{G}\,\pi\,r^2\ . \tag{6.15}$$

  Though u_r' and u_z' are rapidly fluctuating functions with zero mean, they are *correlated* and the mean of their product is not zero. Typically, velocity fluctuations that have $u_r' > 0$ (outwards from the center) also have $u_z' > 0$, because u_z is larger near the centerline.

> **Remark 6.1** *The Reynolds stress should not be thought as a "correction" or a "small perturbation" to an underlying laminar flow. Quite to the contrary, it is the term $\mu\partial_r\bar{w}$ that is negligible througout the flow, with the exception of a narrow layer near the walls.*

6.3 Turbulence models

- If one could express $\boldsymbol{\sigma}^{\mathrm{Re}}$ somehow in terms of $\bar{\mathbf{u}}$ and/or its derivatives, then one could substitute into (6.13) and solve for $\bar{\mathbf{u}}$ and p. This is accomplished by the so-called *Boussinesq turbulent viscosity hypothesis*. It states that

$$\boldsymbol{\sigma}^{\mathrm{Re}} \simeq -\frac{2}{3}\,\rho\,k\,\mathbf{1} + \mu^{\mathrm{t}}\,(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^{T}) \tag{6.16}$$

  where

$$k = \frac{1}{2}\,\overline{\|\mathbf{u}'\|^2} \tag{6.17}$$

  is the *turbulent kinetic energy* (per unit mass) and μ^{t} is the *turbulent viscosity*. This hypothesis agrees with physical observations in many flows, especially if there are no large wakes and if the boundary layer is attached to the wall. The agreement is not perfect in general, but it is sufficient for engineering predictions.

- Prandtl (1904) produced a model for μ^{t} inspired in molecular models of gases. His *mixing length theory* leads to

$$\mu^{\mathrm{t}} = \rho\,\ell^2\,\|\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^{T}\| \tag{6.18}$$

  where ℓ is the so-called Prandtl's mixing length. Again particularizing to the pipe example, it leads to

$$\mu^{\mathrm{t}} = \rho\,\ell^2\,\left|\frac{d\bar{w}}{dr}\right| \ . \tag{6.19}$$

- If y is the distance to the wall, it is intuitive that the length scale of the turbulent vortices, and thus of the mixing, is y itself. In fact, it is fairly accurate that

$$\ell = \kappa\,y \tag{6.20}$$

where κ, the von Karman constant, turns out to be quite universal (all flows, pipes or planes). Beautiful theories have been built that explain this universality and other properties of turbulent flows, we suggest the interest reader to look for the books by Tennekes & Lumley and by Pope.

- For some flows one can follow the more pragmatic approach of Launder & Spalding (1972) and look for empirical expression for ℓ. A vast experience exists on steady flow in circular pipes, from which we can borrow Nikuradse's law:

$$\frac{\ell}{R} = 0.14 - 0.08\left(1-\frac{y}{R}\right)^2 - 0.06\left(1-\frac{y}{R}\right)^4 = 0.14 - 0.08\left(\frac{r}{R}\right)^2 - 0.06\left(\frac{r}{R}\right)^4 \ . \tag{6.21}$$

Exo. 6.2 *The final differential equation is then*

$$\frac{1}{r}\frac{d}{dr}\left((\mu+\mu^t)\,r\,\frac{d\bar{w}}{dr}\right) = \mathcal{G} \tag{6.22}$$

with $\mu^t = \rho\,\ell(r)^2|d\bar{w}/dr|$ *and* $\ell(r)$ *taken from (6.21).*

Numerically solve this equation with appropriate boundary conditions by the finite volume method, with a suitable treatment of the nonlinearity. Plot the resulting velocity profile.

6.4 Wall laws

- Nothing is simple in turbulence. Boundary conditions are no exception. The mean velocity field computed in Exo. 6.2 does not agree with experimental observation.

- The unrealistic prediction can be traced back to the boundary condition $\bar{\mathbf{u}} = 0$ at the wall. The averaged model we have presented so far, being a so-called *high-Reynolds-number* (or *high-Re*) model, is not physically realistic in the close vicinity of the wall, where viscous effects are comparable to (or larger than) turbulent ones. Essentially, we are imposing the boundary condition at a location where the differential equation is not valid.

- The idea is to replace the "natural" condition $\bar{w}(R) = 0$ by some condition at $\tilde{R} < R$, a point within the turbulent-dominated region where (6.22) is valid.

- A popular and frequently accurate boundary condition in CFD is the *logarithmic law of the wall*. Denoting by τ_w the shear stress at the wall, it is customary to define shear velocity $u^* = \sqrt{\tau_w/\rho}$ and then the wall variables (traditionally u is the longitudinal velocity)

$$u^+ = \frac{\bar{w}}{u^*} \ , \qquad y^+ = \frac{y}{\nu/u^*} \ . \tag{6.23}$$

  It so happens that in many turbulent flows, between $y^+ = 20$ and $y^+ = 100$, the following relation holds:

$$u^+ = \frac{1}{\kappa} \ln \ (E\, y^+) \ , \tag{6.24}$$

  where $\kappa \simeq 0.4$ and $E \simeq 9$. How does this provide a boundary condition? A simple way is to *choose* $\tilde{R}$ as satisfying $y^+ = (R - \tilde{R})^+ = 30$. Normally this is a very small correction of the pipe radius, in the micrometer range. For the pipe we considered previously, for example,

$$\tau_w = -\frac{s\,\rho\,g\,D}{4} = -12.25Pa \qquad \Rightarrow \qquad u^* = \sqrt{\frac{|\tau_w|}{\rho}} = 0.11 \text{ m/s} \ . \tag{6.25}$$

As a consequence,

$$(R - \tilde{R})^+ = 30 \qquad \Rightarrow \qquad R - \tilde{R} = 30\frac{\nu}{u^*} = 2.72 \times 10^{-4} \text{ m} \ . \tag{6.26}$$

In the simplified treatment we are following here, we will take $\tilde{R} = R$ and exploit the wall law at $y^+ = 30$ which gives

$$\frac{\bar{w}}{\sqrt{|\tau_w|/\rho}} \ = \ \frac{1}{\kappa} \ln \ (E\, y^+) \ = \ 14 \tag{6.27}$$

so that

$$\left(\mu + \mu^{\mathrm{t}}\right) \ \frac{d\bar{w}}{dr} = - \ \frac{\rho}{196} \ \bar{w}^2 \ , \tag{6.28}$$

which is the boundary condition imposed at $r = R$.

- Notice that we impose a "drag law" and not simply $\left(\mu + \mu^{\mathrm{t}}\right) \ d\bar{w}/dr = \tau_w$, because it is only in very symmetric situations that we know the value of τ_w a priori.

Exo. 6.3 *Compute numerically the velocity profile produced by the model described above. Predict the flow in the pipe of the example, in particular the flow rate and the average velocity. Compute the Reynolds number. Compare to the prediction of the "Moody chart" (google me).*

6.5 Reynolds-averaged Navier-Stokes equations

- In the modeling of incompressible turbulent flows, as we have seen, it is in most cases necessary to solve an averaged version of the Navier-Stokes equations.
- In the pipe-flow example we adopted Nikuradse's formula for the Prandtl's mixing length $\ell(y)$. Such formulae are however only available for some selected flows, in general situations the computation of μ^{t} requires the solution of additional equations.
- There exist 1-equation models, 2-equation models, and so on. Some of the popular ones are known as: Spalart-Allmaras model, $k-\epsilon$ model, $k-\omega$ model, algebraic stress model, stress transport model, etc. An excellent survey is provided by Wilcox (Turbulence modeling for CFD, 2006).
- To provide some insight into RANS modeling, we describe here the $k-\epsilon$ model, which is the most popular 2-equation model.
- Reynolds averaging (substituting $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ in the Navier-Stokes equations and averaging) is the basis of all RANS models:

  Exo. 6.4 *Deduce the RANS equations:*

$$\rho\,\partial_t\bar{\mathbf{u}} + \nabla\cdot\left[\bar{p}\,\mathbb{I} - \mu\left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right) + \rho\,\bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \rho\,\overline{\mathbf{u}'\otimes\mathbf{u}'}\right] = \mathbf{f} \tag{6.29}$$

$$\nabla\cdot\bar{\mathbf{u}} = 0 \tag{6.30}$$

The $k-\epsilon$ model

- The mass and momentum equations are as in the non-averaged case, only that **velocity and pressure variables are now averages** and **turbulent viscosity must be added to physical vis-**

cosity. We drop here the bar to express averages for simplicity.

$$\rho\,\partial_t \mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\;\mathbf{u} - \nabla\;\cdot\;\left[(\mu+\mu^{\mathrm{t}})\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T)\right] + \nabla p \;=\; \mathbf{f} \tag{6.31}$$

$$\nabla\cdot\mathbf{u} \;=\; 0 \tag{6.32}$$

- The turbulent viscosity is given by

$$\mu^{\mathrm{t}} = \frac{c_\mu\,\rho\,k^2}{\epsilon}\;. \tag{6.33}$$

- The turbulent variable k is the **turbulent kinetic energy** per unit mass, the amount of kinetic energy that is contained by the **velocity fluctuations**:

$$k = \frac{1}{2}\,\overline{\|\mathbf{u}'\|^2} \tag{6.34}$$

- The turbulent dissipation ϵ is the **turbulent dissipation rate** per unit mass, the rate at which energy stored in the fluctuations is dissipated:

$$\epsilon = \frac{\mu}{\rho}\,\overline{(\nabla\mathbf{u}'+(\nabla\mathbf{u}')^T):\nabla\mathbf{u}'} \tag{6.35}$$

- The model for k and ϵ consists of the convection-diffusion-reaction equations

$$\partial_t k + \mathbf{u} \cdot \nabla k - \nabla \cdot (D_k \nabla k) + \gamma_k \, k = F_k \tag{6.36}$$

$$\partial_t \epsilon + \mathbf{u} \cdot \nabla \epsilon - \nabla \cdot (D_\epsilon \nabla \epsilon) + \gamma_\epsilon \, \epsilon = F_\epsilon \tag{6.37}$$

where the diffusion coefficients are

$$D_k = \frac{1}{\rho} \left(\frac{\mu^{\mathrm{t}}}{\sigma_k} + \mu \right) , \qquad D_\epsilon = \frac{1}{\rho} \left(\frac{\mu^{\mathrm{t}}}{\sigma_\epsilon} + \mu \right) , \tag{6.38}$$

the reaction coefficients are

$$\gamma_k = \frac{\epsilon}{k} , \qquad \gamma_\epsilon = c_2 \, \frac{\epsilon}{k} , \tag{6.39}$$

and the source terms are

$$F_k = \frac{\mu^{\mathrm{t}}}{2\rho} \, \| \nabla \mathbf{u} + \nabla \mathbf{u}^T \|^2 \, , F_\epsilon = \frac{c_1 k}{2} \, \| \nabla \mathbf{u} + \nabla \mathbf{u}^T \|^2 \, . \tag{6.40}$$

- The model constants have as standard values

$$c_\mu = 0.09 \, , \qquad c_1 = 0.126 \, , \qquad c_2 = 1.92 \, , \qquad \sigma_k = 1.0 \, , \qquad \sigma_\epsilon = 1.3 \, . \tag{6.41}$$

- The closed system of equations of the $k-\epsilon$ model are (6.31), (6.32), (6.36) and (6.37). One vector equation and three scalar ones, for one vector unknown and three scalar unknown.

- The boundary conditions for this model can vary. Most frequent is the *logarithmic law of the wall* as described in Section 6.4. In the simplified treatment adopted there, they would read:

  1. **At inflows:**

$$\mathbf{u} = \mathbf{u}_{\text{in}} \ , \qquad k = k_{\text{in}} \ , \qquad \epsilon = \epsilon_{\text{in}} \ . \tag{6.42}$$

  2. **At planar walls:**

$$\begin{aligned} \mathbf{u} \cdot \check{\mathbf{n}} &= 0 \ , & (6.43) \\ \left(\mu + \mu^{\text{t}}\right) \frac{\partial \mathbf{u}}{\partial n} &= -\frac{\rho}{196} \|\mathbf{u}\|^2 \ , & (6.44) \\ k &= \frac{u^{*2}}{\sqrt{c_\mu}} \ , & (6.45) \\ \epsilon &= \frac{\rho \, u^{*4}}{12.3 \, \mu} \ . & (6.46) \end{aligned}$$

  3. **At outflows:** Several possibilities, zero applied forces and zero normal derivatives of k and ϵ for example.

- The equations for k and ϵ can be written in **conservation form**

$$\partial_t k + \nabla \cdot (\mathbf{u}\, k - D_k \nabla k) = F_k - \gamma_k\, k \;, \tag{6.47}$$

$$\partial_t \epsilon + \nabla \cdot (\mathbf{u}\, \epsilon - D_\epsilon \nabla \epsilon) = F_\epsilon - \gamma_\epsilon\, \epsilon \;, \tag{6.48}$$

  the expressions in parentheses being the **fluxes**.