# Lecture 10 - Turbulence Models

# Applied Computational Fluid Dynamics
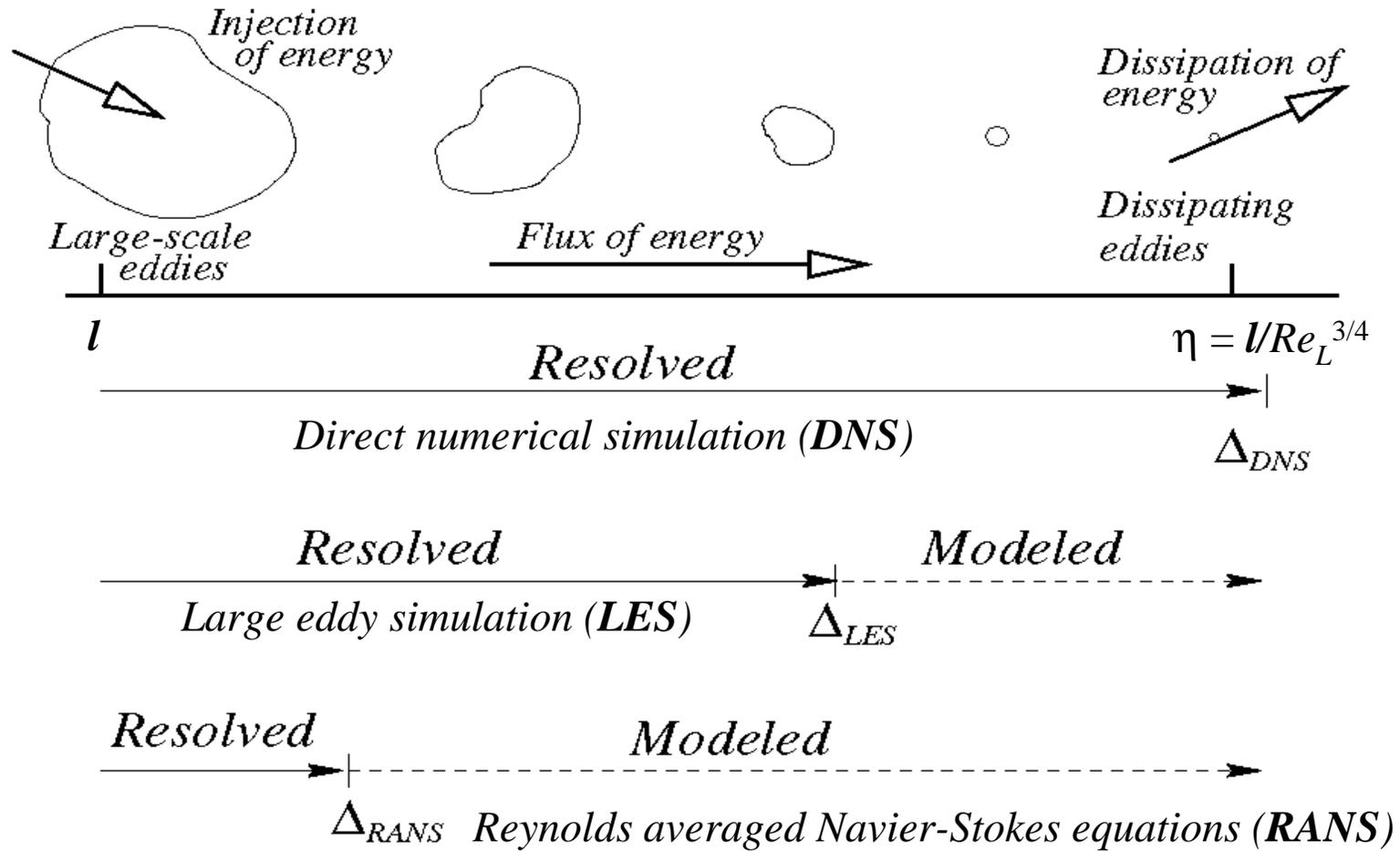
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# Turbulence models

- A turbulence model is a computational procedure to close the system of _mean flow_ equations.
- For most engineering applications it is unnecessary to resolve the details of the turbulent fluctuations.
- Turbulence models allow the calculation of the mean flow without first calculating the full time-dependent flow field.
- We only need to know how turbulence affected the mean flow.
- In particular we need expressions for the Reynolds stresses.
- For a turbulence model to be useful it:
  - must have wide applicability,
  - be accurate,
  - simple,
  - and economical to run.

# Common turbulence models

- Classical models. Based on Reynolds Averaged Navier-Stokes (RANS) equations (time averaged):
  - 1. Zero equation model: mixing length model.
  - 2. One equation model: Spalart-Almaras.
  - 3. Two equation models: $k$-$\varepsilon$ style models (standard, RNG, realizable), $k$-$\omega$ model, and ASM.
  - 4. Seven equation model: Reynolds stress model.
- The number of equations denotes the number of additional PDEs that are being solved.
- Large eddy simulation. Based on space-filtered equations. Time dependent calculations are performed. Large eddies are explicitly calculated. For small eddies, their effect on the flow pattern is taken into account with a "subgrid model" of which many styles are available.

# Prediction Methods



Injection of energy

Dissipation of energy

Large-scale eddies

Flux of energy

Dissipating eddies

$l$

$\eta = l/Re_L^{3/4}$

Resolved

Direct numerical simulation (**DNS**)

$\Delta_{DNS}$

Resolved

Modeled

Large eddy simulation (**LES**)

$\Delta_{LES}$

Resolved

Modeled

$\Delta_{RANS}$ Reynolds averaged Navier-Stokes equations (**RANS**)

# Boussinesq hypothesis

- Many turbulence models are based upon the Boussinesq hypothesis.
  - It was experimentally observed that turbulence decays unless there is shear in isothermal incompressible flows.
  - Turbulence was found to increase as the mean rate of deformation increases.
  - Boussinesq proposed in 1877 that the Reynolds stresses could be linked to the mean rate of deformation.
- Using the suffix notation where i, j, and k denote the x-, y-, and z- directions respectively, viscous stresses are given by:

$$\tau_{ij} = \mu \, e_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

- Similarly, link Reynolds stresses to the mean rate of deformation:

$$\tau_{ij} = -\rho \overline{u_i' u_j'} = \mu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

# Turbulent viscosity

- A new quantity appears: the turbulent viscosity $\mu_t$.

$$\tau_{ij} = -\rho \overline{u_i' u_j'} = \mu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

- Its unit is the same as that of the molecular viscosity: Pa.s.
- It is also called the eddy viscosity.
- We can also define a kinematic turbulent viscosity: $\nu_t = \mu_t/\rho$. Its unit is $m^2/s$.
- The turbulent viscosity is not homogeneous, i.e. it varies in space.
- It is, however, assumed to be isotropic. It is the same in all directions. This assumption is valid for many flows, but not for all (e.g. flows with strong separation or swirl).

# Turbulent Schmidt number

- The turbulent viscosity is used to close the momentum equations.
- We can use a similar assumption for the turbulent fluctuation terms that appear in the scalar transport equations.
- For a scalar property $\phi(t) = \Phi + \varphi'(t)$:

$$-\overline{\rho u_i' \varphi'} = \Gamma_t \frac{\partial \Phi}{\partial x_i}$$

- Here $\Gamma_t$ is the turbulent diffusivity.
- The turbulent diffusivity is calculated from the turbulent viscosity, using a model constant called the turbulent Schmidt number (AKA Prandtl number) $\sigma_t$:
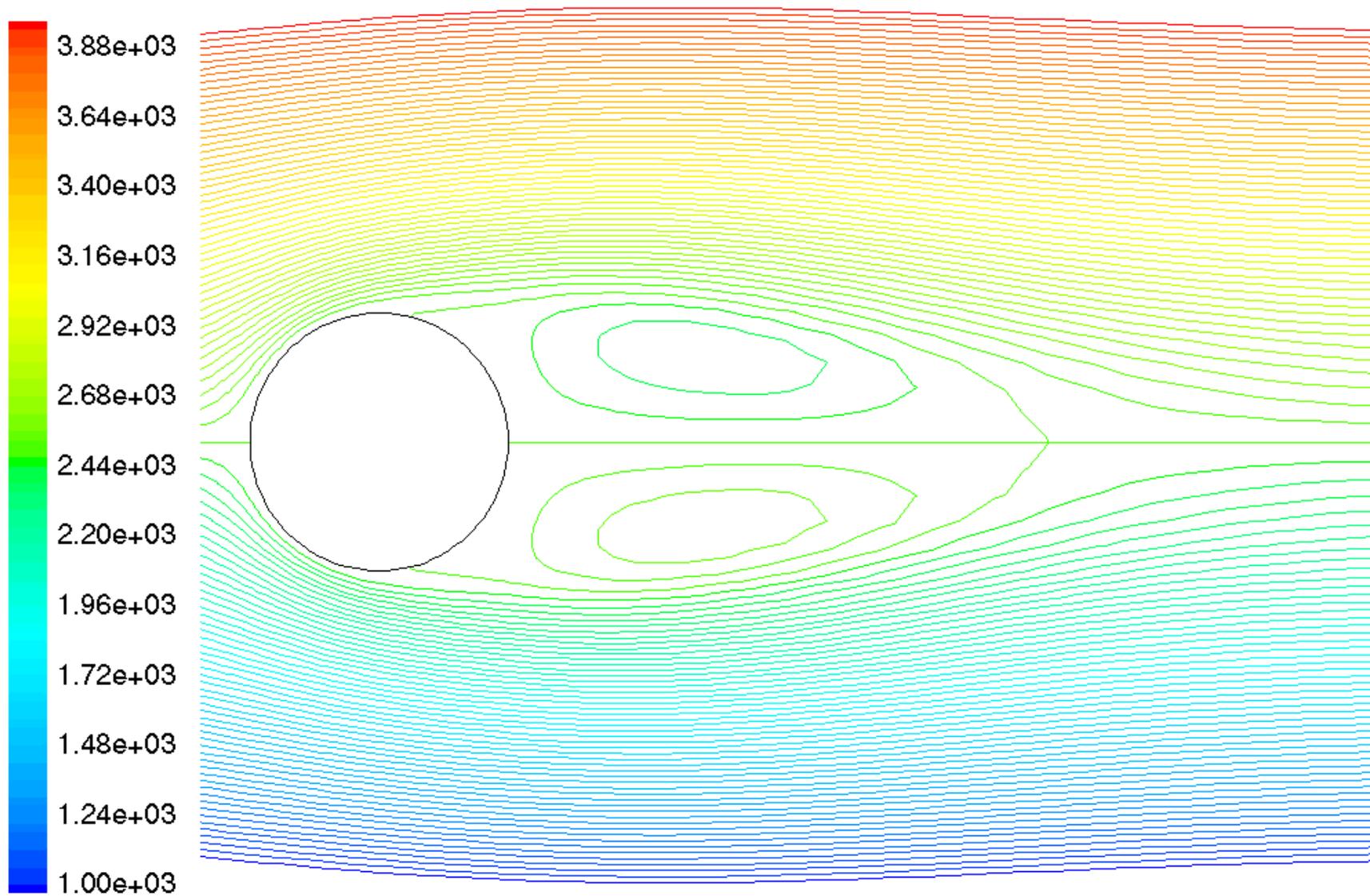
$$\sigma_t = \frac{\mu_t}{\Gamma_t}$$

- Experiments have shown that the turbulent Schmidt number is nearly constant with typical values between 0.7 and 1.
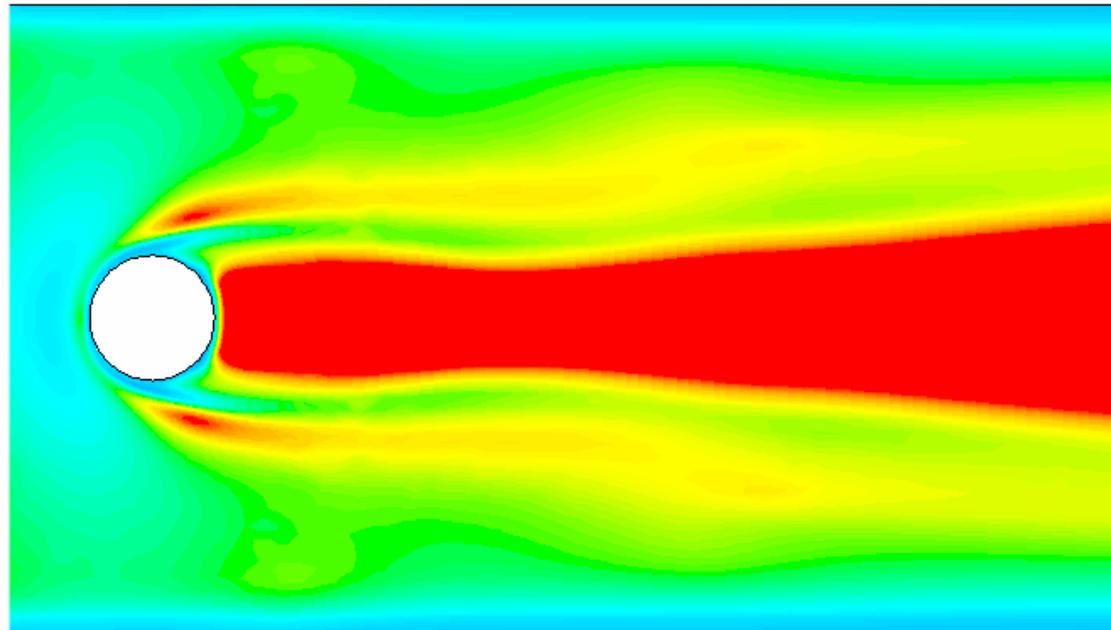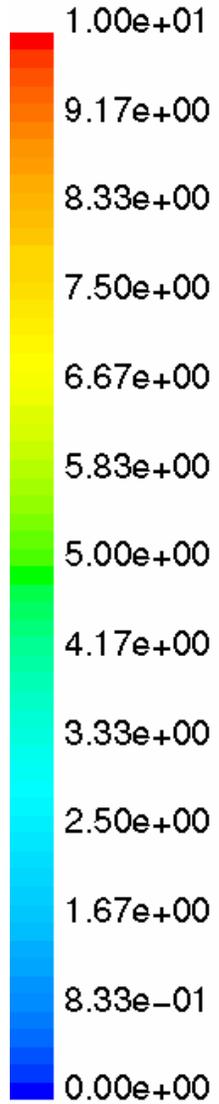
# Flow around a cylinder

- The flow is stable for Reynolds numbers below ~40.

- For higher Reynolds numbers the flow is unstable.
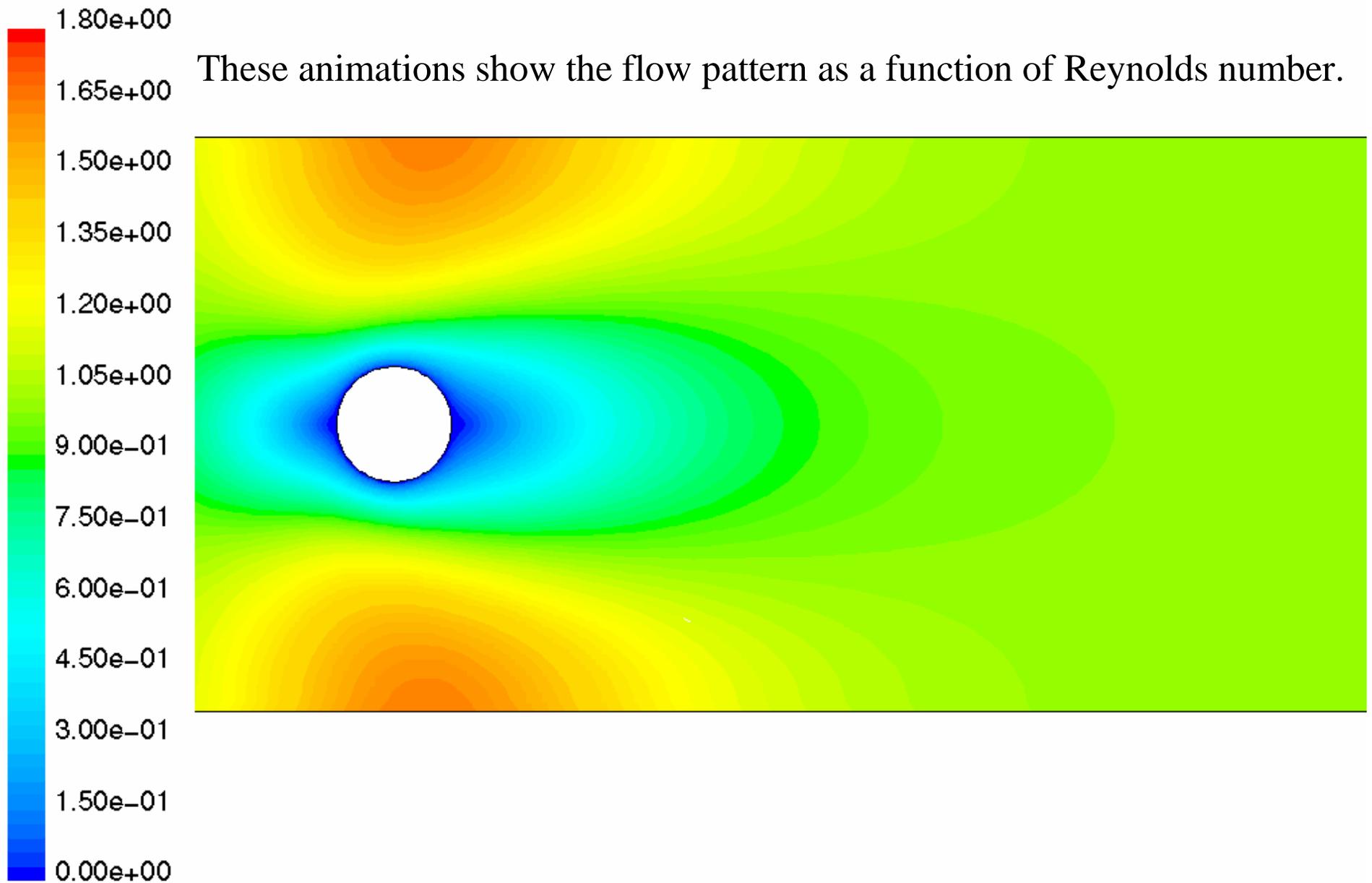
- This animation shows the flow pattern for Re = 1000.
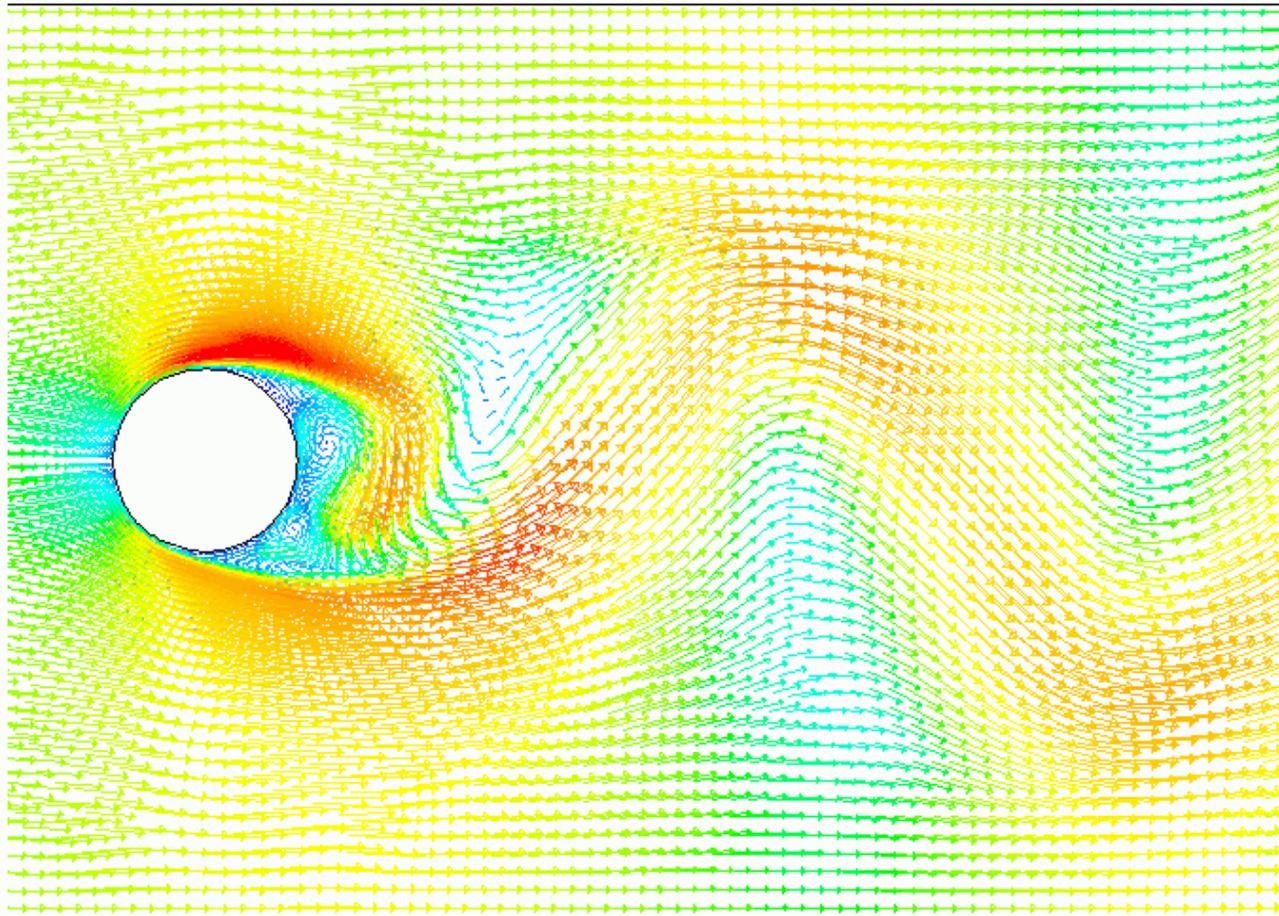
# Time average streamlines (kg/s) - Re = 1000
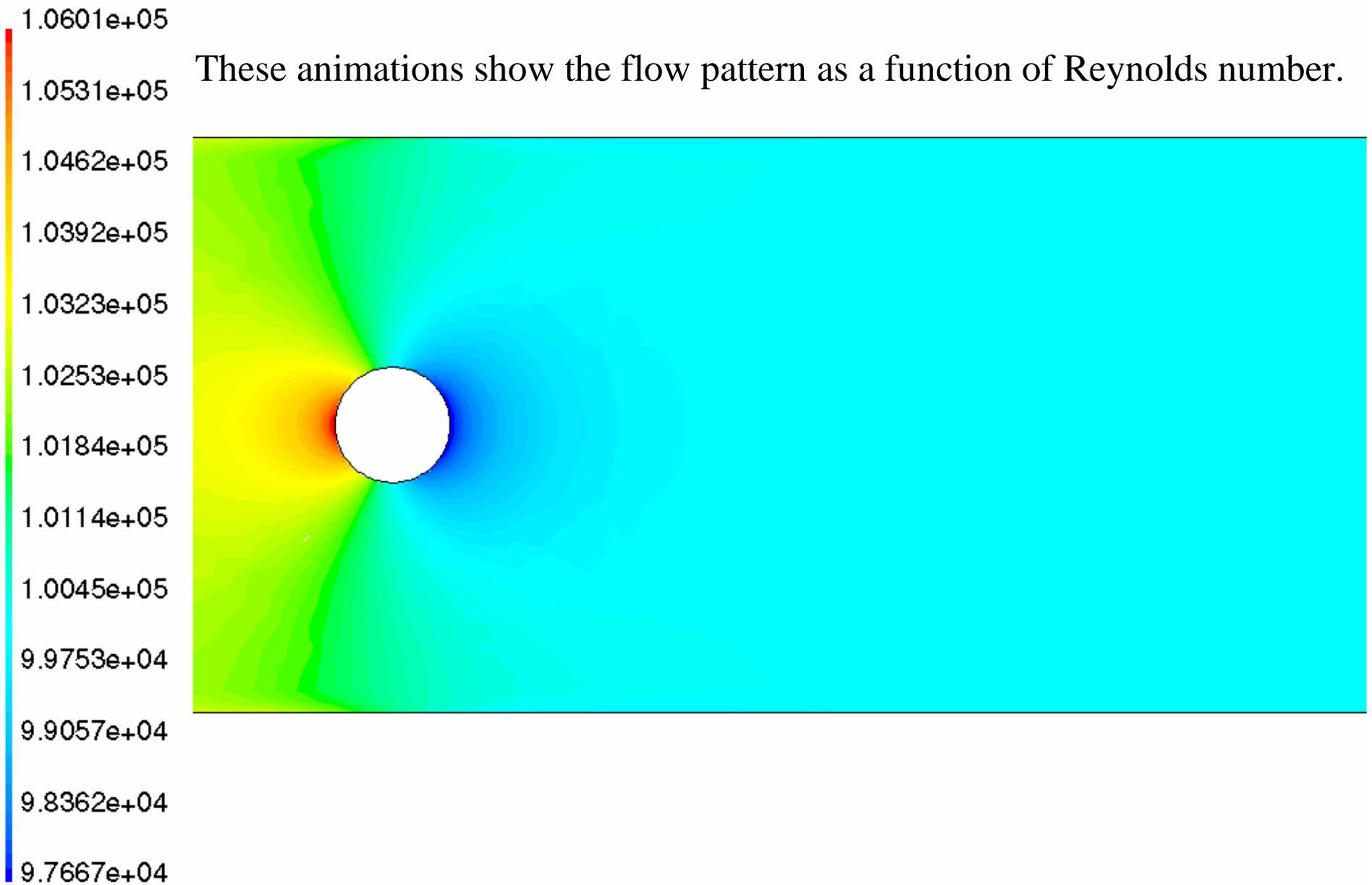
# Effective viscosity (kg/m-s) - Re =1000

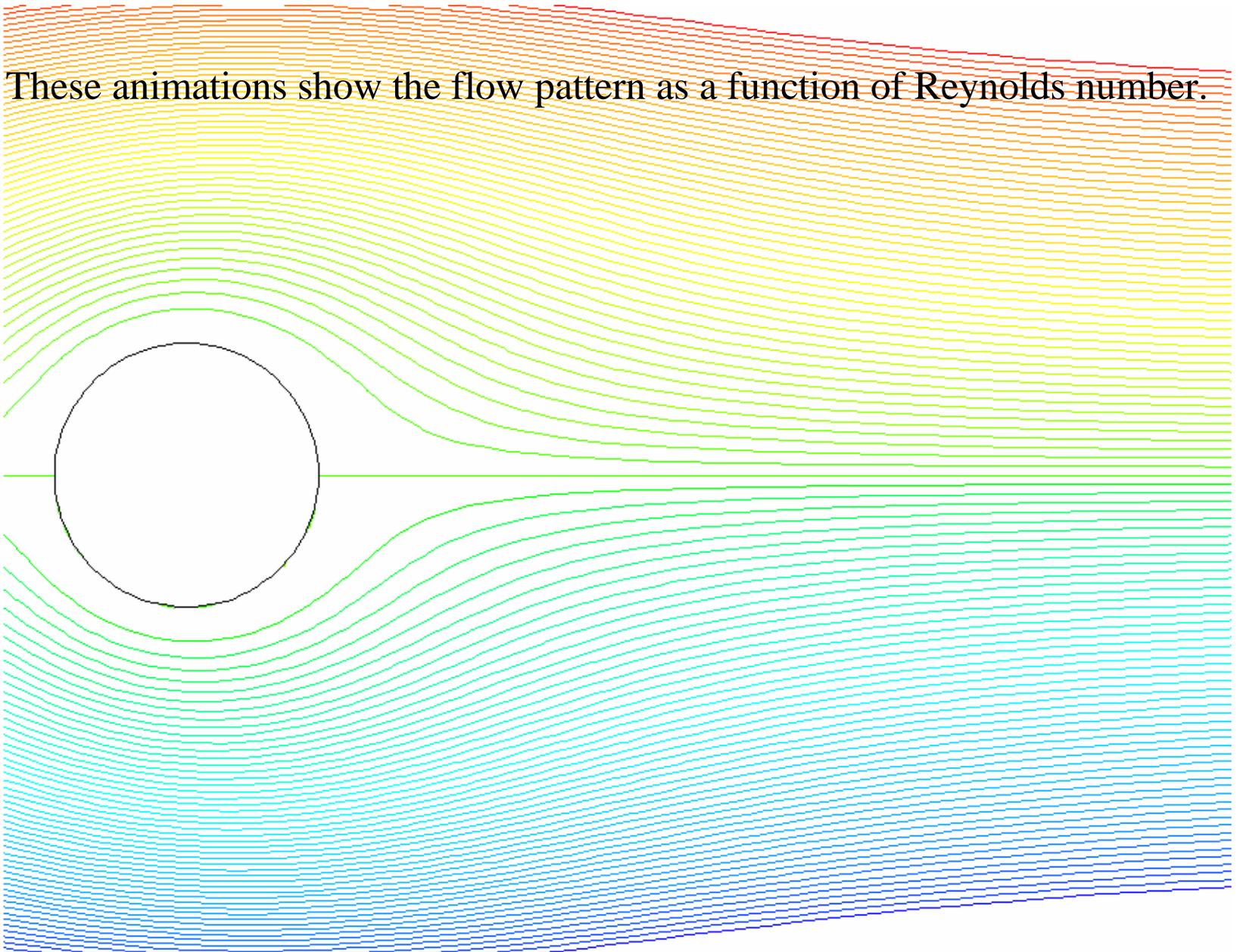These animations show the flow pattern as a function of Reynolds number.

Re=1
Contours of Velocity Magnitude (m/s)
max = 1.65 m/s

These animations show the flow pattern as a function of Reynolds number.

Re=1
Contours of Absolute Pressure (pascal)
(max −min) = 8390 Pa

These animations show the flow pattern as a function of Reynolds number.

Re=1
Contours of Stream Function (kg/s)

These animations show the flow pattern as a function of Reynolds number.
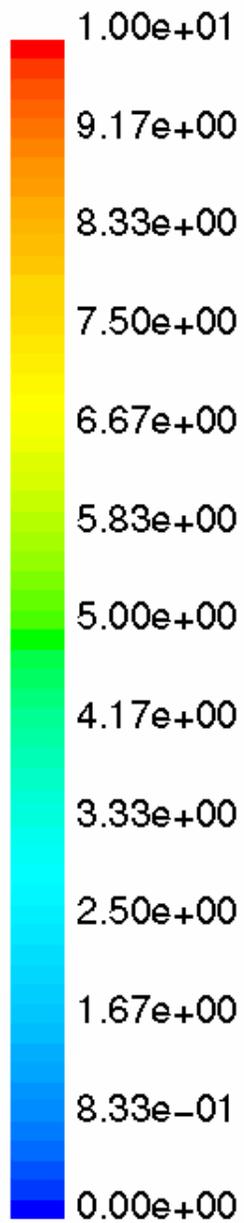
Re = 100
Contours of Molecular Viscosity (kg/m−s)

These animations show the flow pattern as a function of Reynolds number.
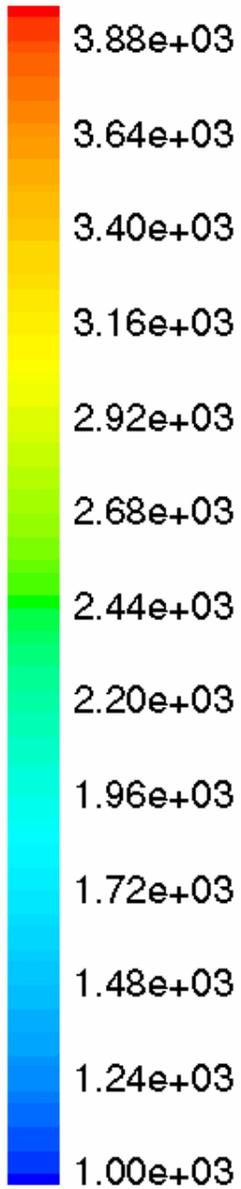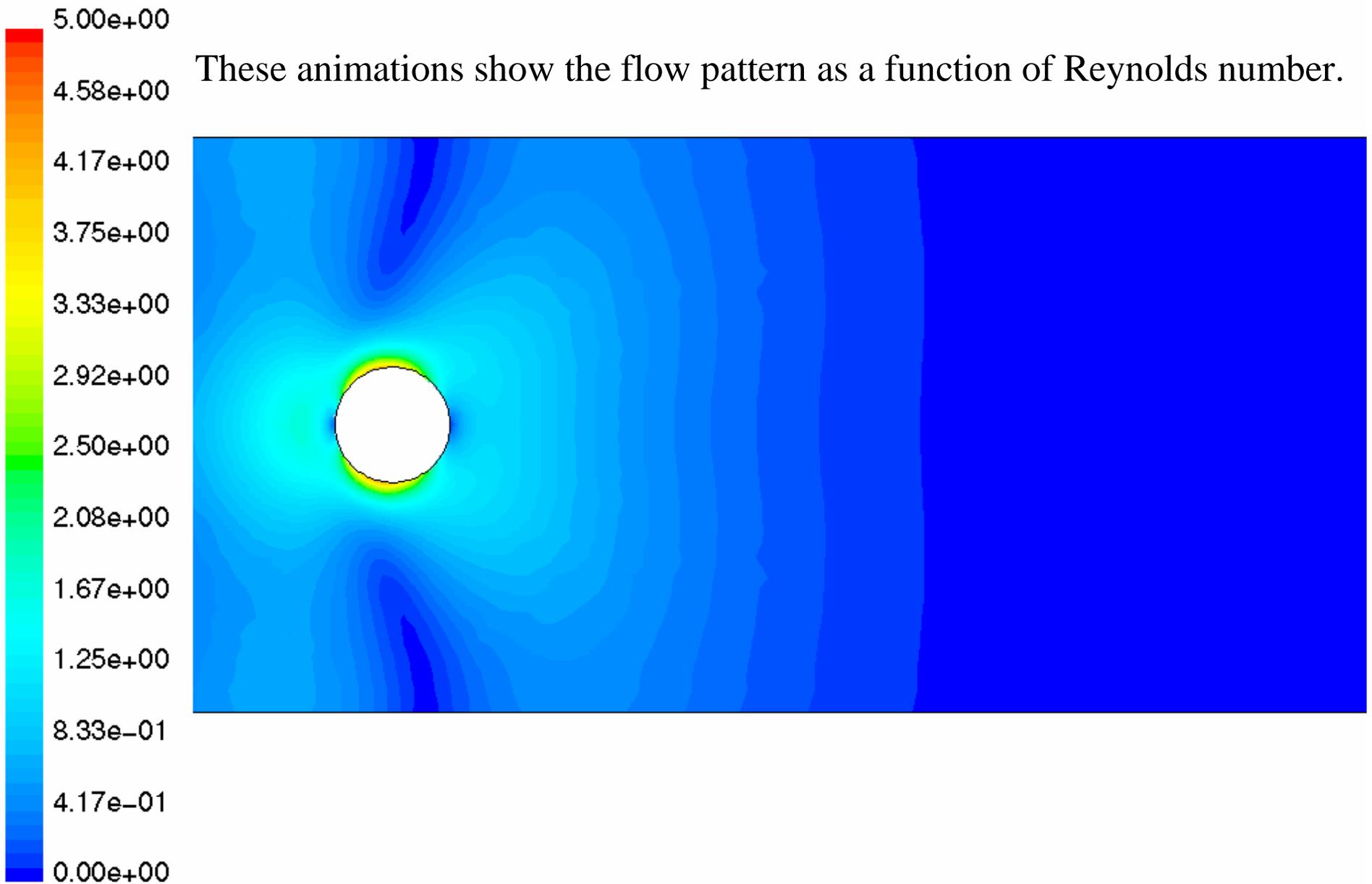
Re = 1
Contours of Strain Rate (1/s)
max = 4.1 1/s

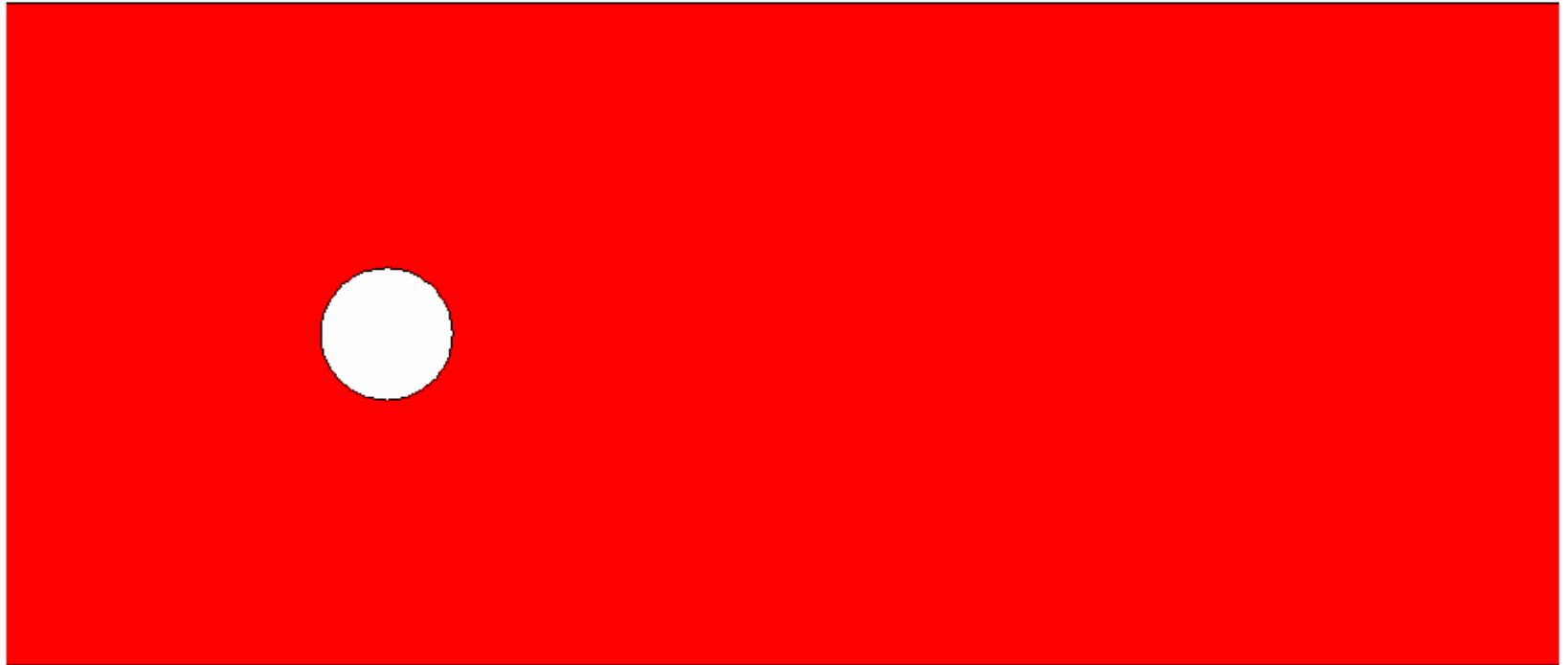These animations show the flow pattern as a function of Reynolds number.
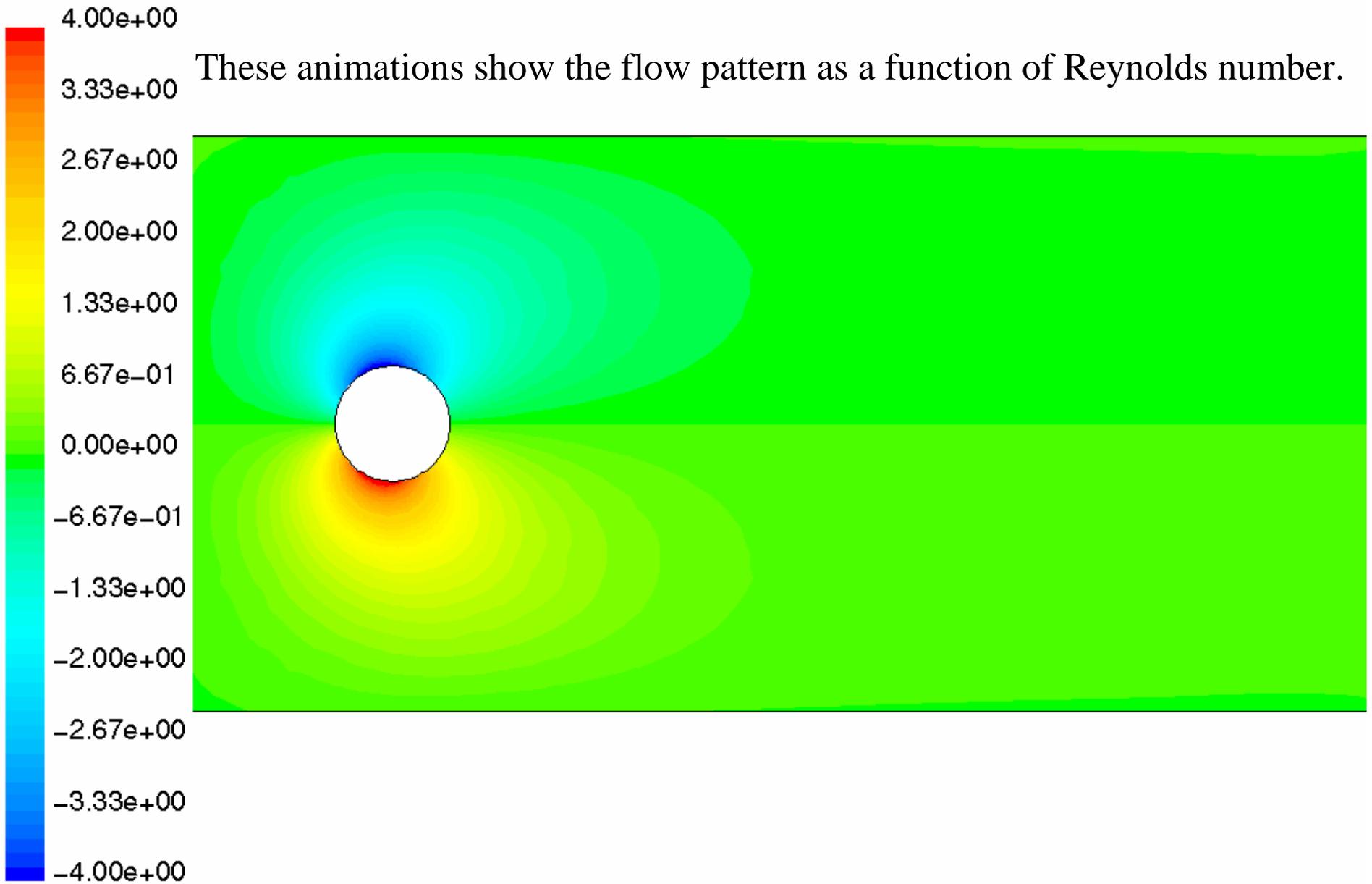
Re = 1
Contours of Vorticity Magnitude

# Predicting the turbulent viscosity

- The following models can be used to predict the turbulent viscosity:
  - Mixing length model.
  - Spalart-Allmaras model.
  - Standard $k$-$\varepsilon$ model.
  - $k$-$\varepsilon$ RNG model.
  - Realizable $k$-$\varepsilon$ model.
  - $k$-$\omega$ model.
- We will discuss these one by one.

# Mixing length model

- On dimensional grounds one can express the kinematic turbulent viscosity as the product of a velocity scale and a length scale:

$$\nu_t \, (m^2/s) \; \propto \; \vartheta(m/s)\,\ell(m)$$

- If we then assume that the velocity scale is proportional to the length scale and the gradients in the velocity (shear rate, which has dimension 1/s):

$$\vartheta \; \propto \; \ell \left| \frac{\partial U}{\partial y} \right|$$

  we can derive Prandtl's (1925) mixing length model:

$$\nu_t \; = \; \ell_m^2 \left| \frac{\partial U}{\partial y} \right|$$

- Algebraic expressions exist for the mixing length for simple 2-D flows, such as pipe and channel flow.

# Mixing length model discussion

- Advantages:
  - Easy to implement.
  - Fast calculation times.
  - Good predictions for simple flows where experimental correlations for the mixing length exist.
- Disadvantages:
  - Completely incapable of describing flows where the turbulent length scale varies: anything with separation or circulation.
  - Only calculates mean flow properties and turbulent shear stress.
- Use:
  - Sometimes used for simple external aero flows.
  - Pretty much completely ignored in commercial CFD programs today.
- Much better models are available.

# Spalart-Allmaras one-equation model

- Solves a single conservation equation (PDE) for the turbulent viscosity:
  - This conservation equation contains convective and diffusive transport terms, as well as expressions for the production and dissipation of $\nu_t$.
  - Developed for use in unstructured codes in the aerospace industry.
- Economical and accurate for:
  - Attached wall-bounded flows.
  - Flows with mild separation and recirculation.
- Weak for:
  - Massively separated flows.
  - Free shear flows.
  - Decaying turbulence.
- Because of its relatively narrow use we will not discuss this model in detail.

# The *k-ε* model

- The *k-ε* model focuses on the mechanisms that affect the turbulent kinetic energy (per unit mass) *k*.

- The instantaneous kinetic energy *k(t)* of a turbulent flow is the sum of mean kinetic energy *K* and turbulent kinetic energy *k*:

$$K = \tfrac{1}{2}\left(U^2 + V^2 + W^2\right)$$

$$k = \tfrac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$k(t) = K + k$$

- ε is the dissipation rate of *k*.

- If *k* and ε are known, we can model the turbulent viscosity as:

$$\nu_t \;\propto\; \vartheta\ell \;\propto\; k^{1/2}\frac{k^{3/2}}{\varepsilon} = \frac{k^2}{\varepsilon}$$

- We now need equations for *k* and ε.

# Mean flow kinetic energy K

- The equation for the mean kinetic energy is as follows:

$$\frac{\partial(\rho K)}{\partial t} + div(\rho K \mathbf{U}) = div\left(-P\mathbf{U} + 2\mu \mathbf{U}E_{ij} - \rho \mathbf{U}\overline{u_i'u_j'}\right) - 2\mu E_{ij}.E_{ij} - (-\rho \overline{u_i'u_j'}.E_{ij})$$

$$(I) \qquad (II) \qquad\qquad (III) \quad (IV) \qquad (V) \qquad\qquad (VI) \qquad\qquad (VII)$$

- Here $E_{ij}$ is the mean rate of deformation tensor.
- This equation can be read as:
  - (I) the rate of change of K, plus
  - (II) transport of K by convection, equals
  - (III) transport of K by pressure, plus
  - (IV) transport of K by viscous stresses, plus
  - (V) transport of K by Reynolds stresses, minus
  - (VI) rate of dissipation of K, minus
  - (VII) turbulence production.

# Turbulent kinetic energy $k$

- The equation for the turbulent kinetic energy $k$ is as follows:

$$\frac{\partial(\rho k)}{\partial t} + div(\rho k \mathbf{U}) = div\left(-\overline{p'\mathbf{u}'} + 2\mu\overline{\mathbf{u}'e_{ij}'} - \rho\tfrac{1}{2}\overline{u_i'.u_i'u_j'}\right) - 2\mu\overline{e_{ij}'.e_{ij}'} + (-\rho\overline{u_i'u_j'}.E_{ij})$$

$$(I) \qquad (II) \qquad\qquad\qquad (III) \quad (IV) \qquad\qquad (V) \qquad\qquad (VI) \qquad\qquad (VII)$$

- Here $e_{ij}$' is fluctuating component of rate of deformation tensor.
- This equation can be read as:
  - (I) the rate of change of $k$, plus
  - (II) transport of $k$ by convection, equals
  - (III) transport of $k$ by pressure, plus
  - (IV) transport of $k$ by viscous stresses, plus
  - (V) transport of $k$ by Reynolds stresses, minus
  - (VI) rate of dissipation of $k$, plus
  - (VII) turbulence production.

# Model equation for *k*

- The equation for *k* contains additional turbulent fluctuation terms, that are unknown. Again using the Boussinesq assumption, these fluctuation terms can be linked to the mean flow.
- The following (simplified) model equation for *k* is commonly used.

$$\frac{\partial(\rho k)}{\partial t} + div(\rho k \mathbf{U}) = div\left[\frac{\mu_t}{\sigma_k} grad\ k\right] + 2\mu_t E_{ij}.E_{ij} - \rho\varepsilon$$

Rate of increase    Convective transport    Diffusive transport    Rate of production    Rate of destruction

- The Prandtl number $\sigma_k$ connects the diffusivity of *k* to the eddy viscosity. Typically a value of 1.0 is used.

# Turbulent dissipation

- The equations look quite similar.

- However, the *k* equation mainly contains primed quantities, indicating that changes in *k* are mainly governed by turbulent interactions.

- Furthermore, term (VII) is equal in both equations. But it is actually negative in the K equation (destruction) and positive in the *k* equation: energy transfers from the mean flow to the turbulence.

- The viscous dissipation term (VI) in the *k* equation $-2\mu \overline{e_{ij}'.e_{ij}'}$ describes the dissipation of *k* because of the work done by the smallest eddies against the viscous stresses.

- We can now define the rate of dissipation per unit mass ε as:

$$\varepsilon = 2\nu \overline{e_{ij}'.e_{ij}'}$$

# Dissipation rate - analytical equation

- The analytical equation for $\varepsilon$ is shown below. Because of the many unknown higher order terms, this equation can not be solved, and simplified model equations need to be derived.

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = - \frac{\partial}{\partial x_k} \left( \nu u_k \overline{\frac{\partial u_i}{\partial x_1} \frac{\partial u_i}{\partial x_1}} + 2 \frac{\nu}{\rho} \overline{\frac{\partial p}{\partial x_i} \frac{\partial u_k}{\partial x_i}} - \nu \frac{\partial \varepsilon}{\partial x_k} \right)$$

$$- 2 \nu \frac{\partial U_i}{\partial x_k} \left( \overline{\frac{\partial u_i}{\partial x_1} \frac{\partial u_k}{\partial x_1}} + \overline{\frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_k}} \right) - 2 \nu u_k \overline{\frac{\partial u_i}{\partial x_1}} \frac{\partial^2 U_i}{\partial x_k \partial x_1}$$

$$- 2 \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_1} \frac{\partial u_k}{\partial x_1}} - 2 \overline{\left( \nu \frac{\partial^2 u_i}{\partial x_k \partial x_1} \right)^2}$$

# Model equation for $\varepsilon$

- A model equation for $\varepsilon$ is derived by multiplying the *k* equation by ($\varepsilon/k$) and introducing model constants.
- The following (simplified) model equation for $\varepsilon$ is commonly used.

$$\underbrace{\frac{\partial(\rho\varepsilon)}{\partial t}}_{\substack{\text{Rate of} \\ \text{increase}}} + \underbrace{div(\rho\varepsilon\mathbf{U})}_{\substack{\text{Convective} \\ \text{transport}}} = \underbrace{div\left[\frac{\mu_t}{\sigma_\varepsilon}grad\ \varepsilon\right]}_{\substack{\text{Diffusive} \\ \text{transport}}} + \underbrace{C_{1\varepsilon}\frac{\varepsilon}{k}2\mu_t E_{ij}.E_{ij}}_{\substack{\text{Rate of} \\ \text{production}}} - \underbrace{C_{2\varepsilon}\rho\frac{\varepsilon^2}{k}}_{\substack{\text{Rate of} \\ \text{destruction}}}$$

- The Prandtl number $\sigma_\varepsilon$ connects the diffusivity of $\varepsilon$ to the eddy viscosity. Typically a value of 1.30 is used.
- Typically values for the model constants $C_{1\varepsilon}$ and $C_{2\varepsilon}$ of 1.44 and 1.92 are used.

# Calculating the Reynolds stresses from *k* and ε

- The turbulent viscosity is calculated from:

$$\mu_t = C_\mu \frac{k^2}{\varepsilon} \qquad\qquad C_\mu = 0.09$$

- The Reynolds stresses are then calculated as follows:

$$-\rho \overline{u_i' u_j'} = \mu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij} = 2\mu_t E_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

$$\delta_{ij} = 1 \quad if \quad i = j \quad and \quad \delta_{ij} = 0 \quad if \quad i \neq j$$

- The *(2/3)ρkδ$_{ij}$* term ensures that the normal stresses sum to *k*.
- Note that the *k*-ε model leads to all normal stresses being equal, which is usually inaccurate.

# *k-ε* model discussion

- Advantages:
  - Relatively simple to implement.
  - Leads to stable calculations that converge relatively easily.
  - Reasonable predictions for many flows.
- Disadvantages:
  - Poor predictions for:
    - swirling and rotating flows,
    - flows with strong separation,
    - axisymmetric jets,
    - certain unconfined flows, and
    - fully developed flows in non-circular ducts.
  - Valid only for fully turbulent flows.
  - Simplistic ε equation.

# More two-equation models

- The $k$-$\varepsilon$ model was developed in the early 1970s. Its strengths as well as its shortcomings are well documented.

- Many attempts have been made to develop two-equation models that improve on the standard $k$-$\varepsilon$ model.

- We will discuss some here:
  - $k$-$\varepsilon$ RNG model.
  - $k$-$\varepsilon$ realizable model.
  - $k$-$\omega$ model.
  - Algebraic stress model.
  - Non-linear models.

# Improvement: RNG *k*- ε

- *k*-ε equations are derived from the application of a rigorous statistical technique (Renormalization Group Method) to the instantaneous Navier-Stokes equations.
- Similar in form to the standard *k*-ε equations but includes:
  - Additional term in ε equation for interaction between turbulence dissipation and mean shear.
  - The effect of swirl on turbulence.
  - Analytical formula for turbulent Prandtl number.
  - Differential formula for effective viscosity.
- Improved predictions for:
  - High streamline curvature and strain rate.
  - Transitional flows.
  - Wall heat and mass transfer.
- But still does not predict the spreading of a round jet correctly.

# RNG *k*-ε equations

- Turbulent kinetic energy:

$$\underbrace{\rho U_i \frac{\partial k}{\partial x_i}}_{\text{Convection}} = \underbrace{\mu_t S^2}_{\text{Generation}} + \underbrace{\frac{\partial}{\partial x_i}\left( \alpha_k \mu_{\text{eff}} \frac{\partial k}{\partial x_i} \right)}_{\text{Diffusion}} - \underbrace{\rho \varepsilon}_{\text{Dissipation}} \qquad \text{where} \quad S \equiv \sqrt{2 S_{ij} S_{ij}}, \quad S_{ij} \equiv \frac{1}{2}\left( \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right)$$

- Dissipation rate:

$$\underbrace{\rho U_i \frac{\partial \varepsilon}{\partial x_i}}_{\text{Convection}} = \underbrace{C_{1\varepsilon}\left( \frac{\varepsilon}{k} \right) \mu_t S^2}_{\text{Generation}} + \underbrace{\frac{\partial}{\partial x_i}\left( \alpha_\varepsilon \mu_{\text{eff}} \frac{\partial \varepsilon}{\partial x_i} \right)}_{\text{Diffusion}} - \underbrace{C_{2\varepsilon}\rho\left( \frac{\varepsilon^2}{k} \right)}_{\text{Destruction}} - \underbrace{R}_{\substack{\text{Additional term}\\ \text{related to mean strain}\\ \text{\& turbulence quantities}}}$$

$\alpha_k, \alpha_\varepsilon, C_{1\varepsilon}, C_{2\varepsilon}$ are derived using RNG theory

- Equations written for steady, incompressible flow without body forces.

# Improvement: realizable *k*-ε

- Shares the same turbulent kinetic energy equation as the standard *k*-ε model.
- Improved equation for ε.
- Variable $C_\mu$ instead of constant.
- Improved performance for flows involving:
  - Planar and round jets (predicts round jet spreading correctly).
  - Boundary layers under strong adverse pressure gradients or separation.
  - Rotation, recirculation.
  - Strong streamline curvature.

# Realizable *k*- ε equations

- Distinctions from standard *k-ε* model:
  - Alternative formulation for turbulent viscosity:

$$\mu_t \equiv \rho C_\mu \frac{k^2}{\varepsilon} \quad \text{where} \quad C_\mu = \frac{1}{A_o + A_s \dfrac{U^* k}{\varepsilon}} \quad \text{is now variable.}$$

  - ($A_0$, $A_s$, and U* are functions of velocity gradients).
  - Ensures positivity of normal stresses: $\overline{u_i^2} \geq 0$
  - Ensures Schwarz's inequality: $(\overline{u_i u_j})^2 \leq \overline{u_i^2}\ \overline{u_j^2}$

  - New transport equation for dissipation rate, ε:

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \rho c_1 S\varepsilon - \rho c_2 \frac{\varepsilon^2}{k + \sqrt{\nu\varepsilon}} + c_{1\varepsilon}\frac{\varepsilon}{k}c_{3\varepsilon}G_b$$

| Diffusion | Generation | Destruction | Buoyancy |

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# Realizable *k*- ε C<sub>μ</sub> equations

- Eddy viscosity computed from.

$$\mu_{\mathrm{t}} = \rho C_\mu \frac{k^2}{\varepsilon}, \quad C_\mu = \frac{1}{A_0 + A_s \dfrac{U^* k}{\varepsilon}}$$

$$U^* \equiv \sqrt{S_{ij} S_{ij} + \Omega_{ij} \Omega_{ij}}$$

$$A_0 = 4.04, \quad A_s = \sqrt{6} \cos \phi, \quad \phi = \frac{1}{3} \cos^{-1}\left(\sqrt{6} W\right)$$

$$W = \frac{S_{ij} S_{ji} S_{ki}}{\tilde{S}}, \quad \tilde{S} = \sqrt{S_{ij} S_{ij}}$$

# Realizable *k*- ε positivity of normal stresses

- Boussinesq viscosity relation:

$$-\rho \, \overline{u_i u_j} = \mu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \rho \, k \, \delta_{ij} \quad ; \quad \mu_t = \rho \, C_\mu \frac{k^2}{\varepsilon}$$

- Normal component:

$$\overline{u^2} = \frac{2}{3} k - 2 C_\mu \frac{k^2}{\varepsilon} \frac{\partial U}{\partial x}$$

- Normal stress will be negative if:

$$\frac{k}{\varepsilon} \frac{\partial U}{\partial x} > \frac{1}{3 C_\mu} \approx 3.7$$

# *k-ω* model

- This is another two equation model. In this model ω is an inverse time scale that is associated with the turbulence.
- This model solves two additional PDEs:
  - A modified version of the *k* equation used in the *k-ε* model.
  - A transport equation for ω.
- The turbulent viscosity is then calculated as follows:

$$\mu_t = \rho \frac{k}{\omega}$$

- Its numerical behavior is similar to that of the *k-ε* models.
- It suffers from some of the same drawbacks, such as the assumption that $\mu_t$ is isotropic.

# Algebraic stress model

- The same $k$ and $\varepsilon$ equations are solved as with the standard $k$-$\varepsilon$ model.

- However, the Boussinesq assumption is not used.

- The full Reynolds stress equations are first derived, and then some simplifying assumptions are made that allow the derivation of algebraic equations for the Reynolds stresses.

- Thus fewer PDEs have to be solved than with the full RSM and it is much easier to implement.

- The algebraic equations themselves are not very stable, however, and computer time is significantly more than with the standard $k$-$\varepsilon$ model.

- This model was used in the 1980s and early 1990s. Research continues but this model is rarely used in industry anymore now that most commercial CFD codes have full RSM implementations available.

# Non-linear models

- The standard *k*-ε model is extended by including second and sometimes third order terms in the equation for the Reynolds stresses.

- One example is the Speziale model:

$$\tau_{ij} = -\rho\overline{u_i{'}u_j{'}} = -\frac{2}{3}k\delta_{ij} + \rho C_\mu \frac{k^2}{\varepsilon}2E_{ij} - 4C_D C_\mu^2 \frac{k^3}{\varepsilon^2} * f(E, \partial E/\partial t, \mathbf{u}, \partial U/\partial x)$$

- Here $f(\ldots)$ is a complex function of the deformation tensor, velocity field and gradients, and the rate of change of the deformation tensor.

- The standard *k*-ε model reduces to a special case of this model for flows with low rates of deformation.

- These models are relatively new and not yet used very widely.

# Reynolds stress model

- RSM closes the Reynolds-Averaged Navier-Stokes equations by solving additional transport equations for the six independent Reynolds stresses.
  - Transport equations derived by Reynolds averaging the product of the momentum equations with a fluctuating property.
  - Closure also requires one equation for turbulent dissipation.
  - Isotropic eddy viscosity assumption is avoided.
- Resulting equations contain terms that need to be modeled.
- RSM is good for accurately predicting complex flows.
  - Accounts for streamline curvature, swirl, rotation and high strain rates.
    - Cyclone flows, swirling combustor flows.
    - Rotating flow passages, secondary flows.
    - Flows involving separation.

# Reynolds stress transport equation

- The exact equation for the transport of the Reynolds stress $R_{ij}$:

$$\frac{DR_{ij}}{Dt} = P_{ij} + D_{ij} - \varepsilon_{ij} + \Pi_{ij} + \Omega_{ij}$$

- This equation can be read as:
  - rate of change of $R_{ij} = \overline{u_i{}' u_j{}'}$ plus
  - transport of $R_{ij}$ by convection, equals
  - rate of production $P_{ij}$, plus
  - transport by diffusion $D_{ij}$, minus
  - rate of dissipation $\varepsilon_{ij}$, plus
  - transport due to turbulent pressure-strain interactions $\pi_{ij}$, plus
  - transport due to rotation $\Omega_{ij}$.

- This equation describes six partial differential equations, one for the transport of each of the six independent Reynolds stresses.

# Reynolds stress transport equation

- The various terms are modeled as follows:
  - Production $P_{ij}$ is retained in its exact form.
  - Diffusive transport $D_{ij}$ is modeled using a gradient diffusion assumption.
  - The dissipation $\varepsilon_{ij}$, is related to $\varepsilon$ as calculated from the standard $\varepsilon$ equation, although more advanced $\varepsilon$ models are available also.
  - Pressure strain interactions $\pi_{ij}$, are very important. These include pressure fluctuations due to eddies interacting with each other, and due to interactions between eddies and regions of the flow with a different mean velocity. The overall effect is to make the normal stresses more isotropic and to decrease shear stresses. It does not change the total turbulent kinetic energy. This is a difficult to model term, and various models are available. Common is the Launder model. Improved, non-equilibrium models are available also.
  - Transport due to rotation $\Omega_{ij}$ is retained in its exact form.

# RSM equations

Production exact : $P_{ij} = -\left( R_{im} \dfrac{\partial U_j}{\partial x_m} + R_{jm} \dfrac{\partial U_i}{\partial x_m} \right)$

Diffusive transport exact : $D_{ij} = \dfrac{\partial J_{ijk}}{\partial x_k}$

$J_{ijk} = \overline{u_i{'}u_j{'}u_k{'}} + \overline{p'(\delta_{jk}u_i{'} + \delta_{ik}u_j{'})}$

Diffusive transport model : $D_{ij} = \dfrac{\partial}{\partial x_m}\left( \dfrac{\nu_t}{\sigma_k} \dfrac{\partial R_{ij}}{\partial x_m} \right) = div\left( \dfrac{\nu_t}{\sigma_k} grad(R_{ij}) \right)$

$\nu_t$ is the turbulent kinematic viscosity calculated in the standard way

# RSM equations continued

Dissipation exact : $\varepsilon_{ij} = 2\mu \overline{\dfrac{\partial u_i'}{\partial x_k} \dfrac{\partial u_j'}{\partial x_k}}$ 

Dissipation model : $\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij}$

Pressure strain exact : $\Pi_{ij} = -\overline{p'\left(\dfrac{\partial u_i'}{\partial x_j} + \dfrac{\partial u_j'}{\partial x_i}\right)}$

Pressure strain model : $\Pi_{ij} = -C_1 \dfrac{\varepsilon}{k}(R_{ij} - \frac{2}{3}k\delta_{ij}) - C_2(P_{ij} - \frac{2}{3}P\delta_{ij})$

$P$ is the pressure

Rotational term (exact) : $\Omega_{ij} = -2\omega_k(R_{jm}e_{ikm} + R_{im}e_{jkm})$

$e_{ijk}$ is $-1, 0$, or $1$ depending on the indices

$\omega_k$ is the rotation vector

# Setting boundary conditions

- Characterize turbulence at inlets and outlets (potential backflow).
  - $k$-$\varepsilon$ models require $k$ and $\varepsilon$.
  - Reynolds stress model requires $R_{ij}$ and $\varepsilon$.
- Other options:
  - Turbulence intensity and length scale.
    - Length scale is related to size of large eddies that contain most of energy.
    - For boundary layer flows, 0.4 times boundary layer thickness:  $l \approx 0.4\delta_{99}$.
    - For flows downstream of grids /perforated plates:  $l \approx$ opening size.
  - Turbulence intensity and hydraulic diameter.
    - Ideally suited for duct and pipe flows.
  - Turbulence intensity and turbulent viscosity ratio.
    - For external flows:  $1 < \mu_t / \mu < 10$

# Comparison of RANS turbulence models

| Model | Strengths | Weaknesses |
|---|---|---|
| Spalart-Allmaras | Economical (1-eq.); good track record for mildly complex B.L. type of flows. | Not very widely tested yet; lack of submodels (e.g. combustion, buoyancy). |
| STD k-$\varepsilon$ | Robust, economical, reasonably accurate; long accumulated performance data. | Mediocre results for complex flows with severe pressure gradients, strong streamline curvature, swirl and rotation. Predicts that round jets spread 15% faster than planar jets whereas in actuality they spread 15% slower. |
| RNG k-$\varepsilon$ | Good for moderately complex behavior like jet impingement, separating flows, swirling flows, and secondary flows. | Subjected to limitations due to isotropic eddy viscosity assumption. Same problem with round jets as standard k-$\varepsilon$. |
| Realizable k-$\varepsilon$ | Offers largely the same benefits as RNG but also resolves the round-jet anomaly. | Subjected to limitations due to isotropic eddy viscosity assumption. |
| Reynolds Stress Model | Physically most complete model (history, transport, and anisotropy of turbulent stresses are all accounted for). | Requires more cpu effort (2-3x); tightly coupled momentum and turbulence equations. |

# Recommendation

- Start calculations by performing 100 iterations or so with standard $k$-$\varepsilon$ model and first order upwind differencing. For very simple flows (no swirl or separation) converge with second order upwind and $k$-$\varepsilon$ model.

- If the flow involves jets, separation, or moderate swirl, converge solution with the realizable $k$-$\varepsilon$ model and second order differencing.

- If the flow is dominated by swirl (e.g. a cyclone or unbaffled stirred vessel) converge solution deeply using RSM and a second order differencing scheme. If the solution will not converge, use first order differencing instead.

- Ignore the existence of mixing length models and the algebraic stress model.

- Only use the other models if you know from other sources that somehow these are especially suitable for your particular problem (e.g. Spalart-Allmaras for certain external flows, $k$-$\varepsilon$ RNG for certain transitional flows, or $k$-$\omega$).