A Minicourse on the Finite Element method - Wrapup

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1 Revision

- The **mesh**: Coordinates + connectivity.

  Coordinates: a matrix that, in each column i (or line?), has the coordinates $\mathbf{X}^i \in \mathbb{R}^d$ of node i.

  Connectivity: a matrix that, in each line i (or column?), has the indices $J_1^i, \ldots, J_{ng}^i$ of the geometrical nodes of element number i.

- The **transformation**: Each element has ng **geometrical basis functions** $\{G_j\}$, and

$$F_K(\hat{x}) = \sum_{i=1}^{ng} \mathbf{X}^{J_j^i} G_j(\hat{x}) \tag{1.1}$$

- The **DOF mesh**: Connectivity of unknowns, i.e. a matrix that, in its line i, has the indices $\overline{J}_1^i, \ldots, \overline{J}_{nd}^i$ of the unknowns of element i.

- The **shape functions**, $\{\widehat{N}_i(\hat{x})\}_{i=1}^{nd}$, defined in $\widehat{K}$.

- The basis functions in each K,

$$N_i(F_K(\hat{x})) = \frac{1}{\alpha_i} \widehat{N}_i(\hat{x}) \tag{1.2}$$

- The functions in the finite element space: Each $x \in \Omega$ (except element borders) is the image $F_K(\hat{x})$ for a unique element K and a unique $\hat{x} \in \widehat{K}$. Given a vector $\underline{U}$ with values for all the DOF of the mesh, the corresponding $u_h \in V_h$ takes the following value at x:

$$u_h(x) = \sum_{j=1}^{nd} U_{\overline{J}_j^K} N_i(x) \ . \tag{1.3}$$

- For FE that are said of class $\mathcal{C}^{\|}$, the resulting u_h has k derivatives continuous at interelement boundaries.

- The space V_h is **parameterized** by $\mathbb{R}^M$ (in fact, isomorphism), where M is the total number of DOFs.

$$u_h = \sum_{j=1}^{M} U_j \, \phi_j \tag{1.4}$$

$$u_h \quad \longleftrightarrow \quad \underline{U}$$

- Refining the mesh or raising the degree of the FE it is possible to approximate with optimal accuracy any function (of some class) with some FE function. It is possible, in particular, to approximate exact solutions of mathematical problems.

- Among others, a **self-adjoint elliptic PDE problems** can be recast as a **minimization problem** over some infinite-dimensional space V of functions defined in Ω.

- An **energy** $E : V \to \mathbb{R}$ is defined, so that the **exact solution** u is the unique minimizer of E. Boundary conditions appear as restrictions or as terms added in E.

$$E(u) \leq E(v) \qquad \forall\, v \in V \tag{1.5}$$

- The **Galerkin FE method** for these problems defines the FE solution u_h as the **unique minimizer** of E over the FE space V_h.

$$E(u_h) \leq E(v_h) \qquad \forall\, v_h \in V_h \tag{1.6}$$

- Thanks to the reparameterization $u_h = \mathcal{U}_h(\underline{U})$, one ends up with a minimization problem in $\mathbb{R}^M$.

$$\mathcal{E}(\underline{U}) \leq \mathcal{E}(\underline{V}) \qquad \forall\, \underline{V} \in \mathbb{R}^M \tag{1.7}$$

  Efficient solvers are available to find the minimum.

- If possible, it is more efficient to provide a code that yields $\{\partial\mathcal{E}/\partial U_i\}$. The DOF vector that makes all partial derivatives be zero is the solution.

- If the energy $\mathcal{E}$ is quadratic (a polynomial of degree two in the unknowns U_i), the underlying mathematical problem is said to be **linear**. The initial guess of the minimization algorithm is not important in linear problems. In nonlinear ones, the minimum may not be found if the initial guess is too far from the solution.

- The obtained solution u_h **converges** to u with **optimal** order of **accuracy** as the FE space is **refined**.

2 Is this all?

- **Standard FE spaces cannot approximate all PDE solutions**: If the coefficients of the PDE are **discontinuous**, and their discontinuities **do not coincide with mesh edges**, then special FE basis functions are needed near the discontinuities.

- **Not all mathematical problems can be recast as minimization problems**.

  - **Non-self-adjoint problems**: Convection-diffusion, Hyperbolic problems, Navier-Stokes, etc.
  - **Time-dependent problems**: Parabolic equations, wave propagation, dynamics, etc.
  - **Eigenvalue problems**: Vibrations, acoustics, etc.

  However, **convergent** FE methods exist for all of them. The justification is based on **variational formulations** instead of the **extremal formulation** that we followed in this minicourse. There exists a version of the **Galerkin method** we discussed for **variational formulations** too.

- **Not all FE methods are Galerkin methods**. Many of the variants are grouped together into the category of **stabilized FE methods**. Explained in terms of what we saw in the minicourse, the idea is to define a **discrete energy** E_h with better properties than the exact energy E. As the mesh is refined, E_h must tend to E in some suitable way so that the minimizer of E_h converges to the minimizer of E.

- Problems with **distributed constraints** such as **incompressibility** require special attention. They lead to the so-called **mixed problems**, for which **mixed finite elements** have been developed.

I hope this was useful.

Thank you very much! ☺