Introduction to the Finite Element method

Gustavo C. Buscaglia

ICMC-USP, São Carlos, Brasil
gustavo.buscaglia@gmail.com

Motivation

- For **elliptic** and **parabolic** problems, the most popular approximation method is the FEM.
- It is **general**, not restricted to linear problems, or to isotropic problems, or to any subclass of mathematical problems.
- It is **geometrically flexible**, complex domains are quite easily treated, not requiring adaptations of the method itself.
- It is **easy to code**, and the coding is quite problem-independent. Boundary conditions are much easier to deal with than in other methods.
- It is **robust**, because in most cases the mathematical problem has an underlying variational structure (energy minimization, for example).

Overview

- **Galerkin approximations:** Differential, variational and extremal formulations of a simple 1D boundary value problem. Well-posedness of variational formulations. Functional setting. Strong and weak coercivity. Lax-Milgram lemma. Banach's open mapping theorem. Céa's best-approximation property. Convergence under weak coercivity. (2 lectures)
- **The spaces of FEM and their implementation:** (3 lectures)
- **Interpolation error and convergence:** (2 lectures)
- **Application to convection-diffusion-reaction problems:** (2 lectures)
- **Application to linear elasticity:** (2 lectures)
- **Mixed problems:** (2 lectures)
- **FEM for parabolic problems:** (2 lectures)

1 Galerkin approximations

1.1 Variational formulation of a simple 1D example

Let u be the solution of

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

The **differential formulation** (DF) of the problem requires $-u''+u$ to be exactly equal to f in **all** points $x \in (0,1)$.

Multiplying the equation by any function v and integrating by parts (recall that

$$\int_0^1 w'\, z \; dx = w(1)z(1) - w(0)z(0) - \int_0^1 w\, z' \; dx \tag{1.2}$$

holds for all w and z that are *regular enough*) one obtains that u satisfies

$$\int_0^1 (u'\, v' + u\, v) \; dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 f\, v \; dx \qquad \forall\, v. \tag{1.3}$$

- The requirement "for all x" of the DF has become "for all functions v".
- Does equation (1.3) fully determine u?
- What happened with the boundary conditions?

Consider the following problem in **variational formulation** (VF): “Determine $u \in W$, such that $u(0) = u(1) = 0$ and that

$$\int_0^1 (u'\,v' + u\,v)\;dx = \int_0^1 f\,v\;dx \tag{1.4}$$

holds for all $v \in W$ satisfying $v(0) = v(1) = 0$.”

Prop. 1.1 *The solution* u *of the DF (eq. 1.1) is also a solution of the VF if* W *consists of continuous functions of sufficient regularity. As a consequence, problem VF admits at least one solution whenever DF does.*

Proof. Following the steps that lead to the VF, it becomes clear that the only requirement for u to satisfy (1.4) is that the integration by parts formula (1.2) be valid. ☐

Exo. 1.1 *Show that the solution of*

$$\begin{cases} -u'' + u = f & \textit{in } (0,1) \\ u(0) = 0, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.5}$$

is a solution to: “Find $u \in W$ *such that* $u(0) = 0$ *and that*

$$\int_0^1 (u'\,v' + u\,v)\;dx = \int_0^1 f\,v\;dx \;+\; g\,v(1) \tag{1.6}$$

holds for all $v \in W$ *satisfying* $v(0) = 0$*.”*

Consider the following problem in **extremal formulation** (EF): "Determine $u \in W$ such that it minimizes the function

$$J(w) = \int_0^1 \left(\frac{1}{2} w'(x)^2 + \frac{1}{2} w(x)^2 - f\, w \right)\, dx \tag{1.7}$$

over the functions $w \in W$ that satisfy $w(0) = w(1) = 0$."

Prop. 1.2 *The unique solution u of (1.1) is also a solution to EF. As a consequence, EF admits at least one solution.*

Proof. We need to show that $J(w) \geq J(u)$ for all $w \in W_0$, where

$$W_0 = \{w \in W\, ,\ w(0) = w(1) = 0\}$$

Writing $w = u + \alpha v$ and replacing in (1.7) one obtains

$$J(u + \alpha\, v) = J(u) + \alpha \left[\int_0^1 (u'\, v' + u\, v - f\, v)\, dx \right] + \alpha^2 \int_0^1 \left(\frac{1}{2} v'(x)^2 + \frac{1}{2} v(x)^2 \right)\, dx$$

The last term is not negative and the second one is zero. □

Exo. 1.2 *Identify the EF of the previous exercise.*

Prop. 1.3 *Let u be the solution of*

$$\begin{cases} -u'' + u = f & \textit{in } (0,1) \\ u(0) = 1, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.8}$$

then u is also a solution of "Determine $u \in W$ such that $u(0) = 1$ and that

$$\int_0^1 (u'\, v' + u\, v)\; dx = \int_0^1 f\, v\; dx \; + \; g\, v(1) \tag{1.9}$$

holds for all $v \in W$ satisfying $v(0) = 0$."
Further, defining for any $a \in \mathbb{R}$

$$W_a = \{w \in W, w(0) = a\},$$

u minimizes over W_1 the function

$$J(w) = \int_0^1 \left(\frac{1}{2} w'(x)^2 + \frac{1}{2} w(x)^2 - f\, w \right)\; dx \; - \; g\, w(1). \tag{1.10}$$

Exo. 1.3 *Prove the last proposition.*

Let us define the bilinear and linear forms corresponding to problem (1.1):

$$a(v,w) = \int_0^1 (v'w' + vw)\, dx \qquad\qquad \ell(v) = \int_0^1 f\, v\, dx \tag{1.11}$$

and the function $J(v) = \frac{1}{2}a(v,v) - \ell(v)$. Remember that W is a space of functions with some (yet unspecified) regularity and let $W_0 = \{w \in W,\ w(0) = w(1) = 0\}$.

The three formulations that we have presented up to now are, thus:

DF: Find a function u such that

$$-u''(x) + u(x) = f(x) \qquad \forall\, x \in (0,1), \qquad\qquad u(0) = u(1) = 0$$

VF: Find a function $u \in W_0$ such that

$$a(u,v) = \ell(v) \;\; \forall\, v \in W_0$$

EF: Find a function $u \in W_0$ such that

$$J(u) \le J(w) \qquad \forall\, w \in W_0$$

and **we know that the exact solution of DF is also a solution of VF and of EF**.

The logic of the construction is justified by the following

Theorem 1.4 *If W is taken as*

$$W = \{w : (0,1) \to \mathbb{R}, \int_0^1 w(x)^2 \; dx < +\infty, \int_0^1 w'(x)^2 \; dx < +\infty\} \overset{\text{def}}{=} H^1(0,1)$$

and if f is such that there exists $C \in \mathbb{R}$ for which

$$\int_0^1 f(x)\, w(x) \; dx \leq C \sqrt{\int_0^1 w'(x)^2 \; dx} \qquad \forall\, w \in W_0 \tag{1.12}$$

then problems (VF) and (EF) have one and only one solution, and their solutions coincide.

The proof will be given later, now let us consider its consequences:

- The differential equation has <u>at most one solution</u> in W.
- If the solution u to (VF)-(EF) is regular enough to be considered a solution to (DF), then u is <u>the solution</u> to (DF).
- If the solution u to (VF)-(EF) is <u>not</u> regular enough to be considered a solution to (DF), then (DF) <u>has no solution</u>.

⇒ (VF) is a <u>generalization</u> of (DF).

Exo. 1.4 *Show that* $W_0 \subset C^0(0,1)$*. Further, compute* $C \in \mathbb{R}$ *such that*

$$\max_{x\in[0,1]} |w(x)| \leq C\sqrt{\int_0^1 w'(x)^2 \; dx} \qquad \forall\, w \in W_0$$

Hint: You may assume that $\int_0^1 f(x)\, g(x) \; dx \leq \sqrt{\int_0^1 f(x)^2 \; dx}\sqrt{\int_0^1 g(x)^2 \; dx}$ *for any* f *and* g *(Cauchy-Schwarz).*

Exo. 1.5 *Consider* $f(x) = |x - 1/2|^\gamma$*. For which exponents* γ *is* $\int_0^1 f(x)\, w(x) \; dx < +\infty$ *for all* $w \in W_0$*?*

Exo. 1.6 *Consider as* f *the "Dirac delta function" at* $x = 1/2$*, that we will denote by* $\delta_{1/2}$*. It can be considered as a "generalized" function defined by*

$$\int_0^1 \delta_{1/2}(x)\, w(x) \; dx = w(1/2) \qquad \forall\, w \in C^0(0,1)$$

Prove that $\delta_{1/2}$ *satisfies (1.12) and determine the analytical solution to (VF).*

Exo. 1.7 *Determine the DF and the EF corresponding to the following VF: "Find* $u \in W = H^1(0,1)$*,* $u(0) = 1$*, such that*

$$\int_0^1 (u'w' + uw) \; dx = w(1/2) \qquad \forall\, w \in W_0 \tag{1.13}$$

where $W_0 = \{w \in W, w(0) = 0\}$*."*

1.2 Variational formulations in general

Let V be a Hilbert space with norm $\| \cdot \|_V$. Let $a(\cdot,\cdot)$ and $\ell(\cdot)$ be bilinear and linear forms on V satisfying (continuity), for all $v,\, w \in V$,

$$a(v,w) \leq N_a \, \|v\|_V \, \|w\|_V, \qquad\qquad \ell(v) \leq N_\ell \, \|v\|_V \tag{1.14}$$

This last inequality means that $\ell \in V'$, the (topological) dual of V. The minimum N_ℓ that satisfies this inequality is called the norm of ℓ in V', i.e.

$$\|\ell\|_{V'} \overset{\text{def}}{=} \sup_{0 \neq v \in V} \frac{\ell(v)}{\|v\|_V} \tag{1.15}$$

The abstract VF we consider here is:

$$\text{“Find } u \in V \text{ such that} \qquad a(u,v) = \ell(v) \qquad\qquad \forall\, v \in V\text{”} \tag{1.16}$$

Exo. 1.8 *Assume that V is finite dimensional, of dimension n, and let $\{\phi^1, \phi^2, \ldots, \phi^n\}$ be a basis. Show that (1.16) is then equivalent to*

$$\underline{V}^T \, \underline{\underline{A}} \, \underline{U} = \underline{V}^T \, \underline{L} \qquad\qquad \forall \, \underline{V} \in \mathbb{R}^n \ , \tag{1.17}$$

which in turn is equivalent to the linear system

$$\underline{\underline{A}} \ \underline{U} = \underline{L} \ ; \tag{1.18}$$

where

$$A_{ij} \overset{\text{def}}{=} a(\phi^j, \phi^i), \qquad L_i \overset{\text{def}}{=} \ell(\phi^i) \tag{1.19}$$

and $\underline{U}$ is the coefficient column vector of the expansion of u, i.e.,

$$u = \sum_{i=1}^{n} U_i \, \phi^i \tag{1.20}$$

Def. 1.5 *The bilinear form* $a(\cdot,\cdot)$ *is said to be* **strongly coercive** *if there exists* $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall\, v \,\in\, V \tag{1.21}$$

Def. 1.6 *The bilinear form* $a(\cdot,\cdot)$ *is said to be* **weakly coercive** *(or to satisfy an* **inf-sup** *condition) if there exists* $\beta > 0$ *such that*

$$\sup_{0\neq w\in V} \frac{a(v,w)}{\|w\|_V} \geq \beta \|v\|_V \qquad \forall\, v \,\in\, V \tag{1.22}$$

and

$$\sup_{0\neq v\in V} \frac{a(v,w)}{\|v\|_V} \geq \beta \|w\|_V \qquad \forall\, w \,\in\, V \tag{1.23}$$

Exo. 1.9 *Prove that strong coercivity implies weak coercivity.*

Exo. 1.10 *Prove that, if* V *is finite dimensional, then* **(i)** $a(\cdot,\cdot)$ *is strongly coercive iff* $\underline{\underline{A}}$ *is positive definite* $(\underline{X}^T\ \underline{\underline{A}}\ \underline{X} > 0\ \forall\, \underline{X}\ \in \mathbb{R}^n)$*, and* **(ii)** $a(\cdot,\cdot)$ *is weakly coercive iff* $\underline{\underline{A}}$ *is invertible.*

Exo. 1.11 *Prove that, if* $a(\cdot,\cdot)$ *is weakly coercive, then the solution* u *of (1.16) depends continuously on the forcing* $\ell(\cdot)$*. Specifically, prove that*

$$\|u\|_V \leq \frac{1}{\beta}\ \|\ell\|_{V'} \tag{1.24}$$

Theorem 1.7 *Assuming V to be a Hilbert space, problem (1.16) is well posed for any $\ell \in V'$ if and only if* **(i)** *$a(\cdot,\cdot)$ is continuous, and* **(ii)** *$a(\cdot,\cdot)$ is weakly coercive.*

A simpler version of this result is known as **Lax-Milgram lemma**:

Theorem 1.8 *Assuming V to be a Hilbert space, if $a(\cdot,\cdot)$ is continuous and strongly coercive then problem (1.16) is well posed for any $\ell \in V'$.*

Proof. This proof uses the so-called "Galerkin method", which will be useful to introduce... the Galerkin method!
Let $\{\phi^i\}$ be a basis of V. Denoting $V_N = \text{span}(\phi^1, \dots, \phi^N)$ we can define $u_N \in V_N$ as the unique solution of $a(u_N, v) = \ell(v)$ for all $v \in V_N$. This generates a sequence $\{u_N\}_{N=1,2,\dots}$ in V. Further, this sequence is bounded, because

$$\|u_N\|_V^2 \le \frac{1}{\alpha}\, a(u_N, u_N) = \frac{1}{\alpha}\, \ell(u_N) \le \frac{\|\ell\|_{V'}}{\alpha}\, \|u_N\|_V \quad \Rightarrow \quad \|u_N\|_V \le \frac{\|\ell\|_{V'}}{\alpha}, \quad \forall N$$

Recalling the <u>weak compactness</u> of bounded sets in Hilbert spaces, there exists $u \in V$ such that a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$ for simplicity) converges to u weakly. It remains to prove that $a(u, v) = \ell(v)$ for all $v \in V$. To see this, notice that

$$a(u, \phi^i) = a(\lim_N u_N, \phi^i) = \lim_N a(u_N, \phi^i) = \ell(\phi^i)$$

where the last equality holds because $a(u_N, \phi^i) = \ell(\phi^i)$ whenever $N \ge i$. Uniqueness is left as an exercise. □

Exo. 1.12 *Prove uniqueness in the previous theorem (bounded sequences may have several accumulation points).*

Remark 1.9 *The space $L^2(a,b)$ (also denoted by $H^0(a,b)$) is the* Hilbert space *of functions $f:(a,b)\to\mathbb{R}$ such that $\int_a^b f^2(x)\,dx < +\infty$.*
The scalar product is

$$(f,g)_{L^2(a,b)} = \int_a^b f(x)g(x)\ dx \tag{1.25}$$

and accordingly

$$\|f\|_{L^2(a,b)} = (f,f)^{1/2}_{L^2(a,b)} = \sqrt{\int_a^b f^2(x)\ dx}\ . \tag{1.26}$$

Also of frequent use are the Hilbert spaces $H^1(a,b)$ and $H^2(a,b)$:

$$\begin{aligned}
H^1(a,b) &= \{f \in L^2(a,b) \mid f' \in L^2(a,b)\} && (1.27)\\
|f|_{H^1(a,b)} &= \|f'\|_{L^2(a,b)} && (1.28)\\
\|f\|_{H^1(a,b)} &= \|f\|_{L^2(a,b)} + |f|_{H^1(a,b)} && (1.29)\\
H^2(a,b) &= \{f \in H^1(a,b) \mid f'' \in L^2(a,b)\} && (1.30)\\
|f|_{H^2(a,b)} &= \|f''\|_{L^2(a,b)} && (1.31)\\
\|f\|_{H^2(a,b)} &= \|f\|_{H^1(a,b)} + |f|_{H^2(a,b)} && (1.32)
\end{aligned}$$

Exo. 1.13 *Other* equivalent *norms can be defined in $H^1(a,b)$, e.g.,*

1. $|||f|||_{H^1(a,b)} = \left(\|f\|^2_{L^2(a,b)} + |f|^2_{H^1(a,b)}\right)^{1/2}$
2. $|||f|||_{H^1(a,b)} = \max\left(\|f\|_{L^2(a,b)}, |f|_{H^1(a,b)}\right)$
3. $|||f|||_{H^1(a,b)} = \|f\|_{L^2(a,b)} + \|\ell\, f'\|_{L^2(a,b)}$*, where $\ell:(a,b)\to\mathbb{R}$ satisfies $0<\ell_{\min}\le\ell(x)\le\ell_{\max}$ for all $x\in(a,b)$. Notice that if $\ell(x)$ has dimensions of length then this norm is unit-consistent.*

Find the constants c and C such that $c\|f\| \leq |||f||| \leq C\|f\|$.

Remark 1.10 *For the spaces $H^1(a,b)$ and $H^2(a,b)$ to be complete, one needs a weaker definition of the derivative. For this purpose, one first introduces the space*

$$\mathcal{D}(a,b) = C_0^\infty(a,b) = \{\varphi \in C^\infty(a,b) \mid \varphi \text{ has compact support in } (a,b)\} \ . \tag{1.33}$$

Given a function $f : (a,b) \to \mathbb{R}$, if there exists $g : (a,b) \to \mathbb{R}$ such that

$$\int_a^b g(x)\,\varphi(x)\ dx = -\int_a^b f(x)\,\varphi'(x)\ dx\ , \qquad \forall\,\varphi \in \mathcal{D}(a,b)\ , \tag{1.34}$$

then we say that f' exists **in a weak sense**, *and that $f' = g$.*

Exo. 1.14 *Show that the function*

$$\phi(x) = \begin{cases} \exp\left(1/(|x|^2 - 1)\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{1.35}$$

belongs to $\mathcal{D}(\mathbb{R})$. By suitably shifting and scaling the argument of ϕ show that $\mathcal{D}(a,b)$ has infinite dimension for all $a < b$. (Hint: See Brenner-Scott, p. 27)

Exo. 1.15 *Consider $f(x) = 1 - |x|$ in the domain $(-1,1)$. Prove that its weak derivative is given by*

$$f'(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases} \ . \tag{1.36}$$

Prove also that f'' does not exist. (Hint: See Brenner-Scott, p. 28)

Exo. 1.16 *Let $f \in L^2(a,b)$, and let $V = H^1(a,b)$. Show that $\ell(v) = \int_a^b f(x)\,v(x)\,dx$ belongs to V' and that $\|\ell\|_{V'} \leq \|f\|_{L^2(a,b)}$.*