Introduction to the Finite Element method

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Motivation

- For **elliptic** and **parabolic** problems, the most popular approximation method is the FEM.
- It is **general**, not restricted to linear problems, or to isotropic problems, or to any subclass of mathematical problems.
- It is **geometrically flexible**, complex domains are quite easily treated, not requiring adaptations of the method itself.
- It is **easy to code**, and the coding is quite problem-independent. Boundary conditions are much easier to deal with than in other methods.
- It is **robust**, because in most cases the mathematical problem has an underlying variational structure (energy minimization, for example).

Overview

- **Galerkin approximations:** Differential, variational and extremal formulations of a simple 1D boundary value problem. Well-posedness of variational formulations. Functional setting. Strong and weak coercivity. Lax-Milgram lemma. Banach's open mapping theorem. Céa's best-approximation property. Convergence under weak coercivity. (2 lectures)
- **The spaces of FEM:** (3 lectures)
- **Interpolation error and convergence:** (1 lecture)
- **Application to convection-diffusion-reaction problems:** (2 lectures)
- **Application to linear elasticity:** (1 lecture)
- **Mixed problems:** (2 lectures)
- **FEM for parabolic problems:** (2 lectures)

1 Galerkin approximations

1.1 Variational formulation of a simple 1D example

Let u be the solution of

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

The **differential formulation** (DF) of the problem requires $-u'' + u$ to be exactly equal to f in **all** points $x \in (0,1)$.
Multiplying the equation by any function v and integrating by parts (recall that

$$\int_0^1 w'\, z\; dx = w(1)z(1) - w(0)z(0) - \int_0^1 w\, z'\; dx \tag{1.2}$$

holds for all w and z that are *regular enough*) one obtains that u satisfies

$$\int_0^1 (u'\, v' + u\, v)\; dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 f\, v\; dx \qquad \forall\, v. \tag{1.3}$$

- The requirement "for all x" of the DF has become "for all functions v".
- Does equation (1.3) fully determine u?
- What happened with the boundary conditions?

Consider the following problem in **variational formulation** (VF): "Determine $u \in W$, such that $u(0) = u(1) = 0$ and that

$$\int_0^1 (u'\,v' + u\,v)\; dx = \int_0^1 f\,v\; dx \tag{1.4}$$

holds for all $v \in W$ satisfying $v(0) = v(1) = 0$."

Prop. 1.1 *The solution u of the DF (eq. 1.1) is also a solution of the VF if W consists of continuous functions of sufficient regularity. As a consequence, problem VF admits at least one solution whenever DF does.*

Proof. Following the steps that lead to the VF, it becomes clear that the only requirement for u to satisfy (1.4) is that the integration by parts formula (1.2) be valid. $\square$

Exo. 1.1 *Show that the solution of*

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = 0, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.5}$$

is a solution to: "Find $u \in W$ such that $u(0) = 0$ and that

$$\int_0^1 (u'\,v' + u\,v)\; dx = \int_0^1 f\,v\; dx \; + \; g\,v(1) \tag{1.6}$$

holds for all $v \in W$ satisfying $v(0) = 0$."

Consider the following problem in **extremal formulation** (EF): "Determine $u \in W$ such that it minimizes the function

$$J(w) = \int_0^1 \left(\frac{1}{2} w'(x)^2 + \frac{1}{2} w(x)^2 - f\, w \right)\, dx \tag{1.7}$$

over the functions $w \in W$ that satisfy $w(0) = w(1) = 0$."

Prop. 1.2 *The unique solution u of (1.1) is also a solution to EF. As a consequence, EF admits at least one solution.*

Proof. We need to show that $J(w) \geq J(u)$ for all $w \in W_0$, where

$$W_0 = \{w \in W \,,\, w(0) = w(1) = 0\}$$

Writing $w = u + \alpha v$ and replacing in (1.7) one obtains

$$J(u + \alpha\, v) = J(u) + \alpha \left[\int_0^1 (u'\, v' + u\, v - f\, v)\, dx \right] + \alpha^2 \int_0^1 \left(\frac{1}{2} v'(x)^2 + \frac{1}{2} v(x)^2 \right)\, dx$$

The last term is not negative and the second one is zero. $\square$

Exo. 1.2 *Identify the EF of the previous exercise.*

Prop. 1.3 *Let u be the solution of*

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = 1, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.8}$$

then u is also a solution of "Determine $u \in W$ such that $u(0) = 1$ and that

$$\int_0^1 (u'\,v' + u\,v)\;dx = \int_0^1 f\,v\;dx \;+\; g\,v(1) \tag{1.9}$$

holds for all $v \in W$ satisfying $v(0) = 0$."
Further, defining for any $a \in \mathbb{R}$

$$W_a = \{w \in W, w(0) = a\},$$

u minimizes over W_1 the function

$$J(w) = \int_0^1 \left(\frac{1}{2}w'(x)^2 + \frac{1}{2}w(x)^2 - f\,w\right)\;dx \;-\; g\,w(1). \tag{1.10}$$

Exo. 1.3 *Prove the last proposition.*

Let us define the bilinear and linear forms corresponding to problem (1.1):

$$a(v,w) = \int_0^1 (v'w' + vw)\; dx \qquad\qquad \ell(v) = \int_0^1 f\, v\; dx \tag{1.11}$$

and the function $J(v) = \frac{1}{2}a(v,v) - \ell(v)$. Remember that W is a space of functions with some (yet unspecified) regularity and let $W_0 = \{w \in W,\, w(0) = w(1) = 0\}$.

The three formulations that we have presented up to now are, thus:

DF: Find a function u such that

$$-u''(x) + u(x) = f(x) \qquad \forall\, x \in (0,1), \qquad\qquad u(0) = u(1) = 0$$

VF: Find a function $u \in W_0$ such that

$$a(u,v) = \ell(v) \;\; \forall\, v \in W_0$$

EF: Find a function $u \in W_0$ such that

$$J(u) \le J(w) \qquad \forall\, w \in W_0$$

and **we know that the exact solution of DF is also a solution of VF and of EF**.

The logic of the construction is justified by the following

Theorem 1.4 *If W is taken as*

$$W = \{w : (0,1) \to \mathbb{R}, \int_0^1 w(x)^2 \, dx < +\infty, \int_0^1 w'(x)^2 \, dx < +\infty\} \stackrel{\text{def}}{=} H^1(0,1)$$

and if f is such that there exists $C \in \mathbb{R}$ for which

$$\int_0^1 f(x)\, w(x) \, dx \leq C \sqrt{\int_0^1 w'(x)^2 \, dx} \qquad \forall\, w \in W_0 \tag{1.12}$$

then problems (VF) and (EF) have one and only one solution, and their solutions coincide.

The proof will be given later, now let us consider its consequences:

- The differential equation has <u>at most one solution</u> in W.
- If the solution u to (VF)-(EF) is regular enough to be considered a solution to (DF), then u is <u>the solution</u> to (DF).
- If the solution u to (VF)-(EF) is <u>not</u> regular enough to be considered a solution to (DF), then (DF) <u>has no solution</u>.

$\Rightarrow$ (VF) is a <u>generalization</u> of (DF).

Exo. 1.4 *Show that $W_0 \subset C^0(0,1)$. Further, compute $C \in \mathbb{R}$ such that*

$$\max_{x\in[0,1]} |w(x)| \leq C\sqrt{\int_0^1 w'(x)^2\ dx} \qquad \forall\, w\, \in\, W_0$$

Hint: You may assume that $\int_0^1 f(x)\, g(x)\ dx \leq \sqrt{\int_0^1 f(x)^2\ dx}\sqrt{\int_0^1 g(x)^2\ dx}$ for any f and g (Cauchy-Schwarz).

Exo. 1.5 *Consider $f(x) = |x - 1/2|^\gamma$. For which exponents γ is $\int_0^1 f(x)\, w(x)\ dx < +\infty$ for all $w\, \in\, W_0$?*

Exo. 1.6 *Consider as f the "Dirac delta function" at $x = 1/2$, that we will denote by $\delta_{1/2}$. It can be considered as a "generalized" function defined by*

$$\int_0^1 \delta_{1/2}(x)\, w(x)\ dx = w(1/2) \qquad \forall\, w\, \in\, C^0(0,1)$$

Prove that $\delta_{1/2}$ satisfies (1.12) and determine the analytical solution to (VF).

Exo. 1.7 *Determine the DF and the EF corresponding to the following VF: "Find $u\, \in\, W = H^1(0,1)$, $u(0) = 1$, such that*

$$\int_0^1 (u'w' + uw)\ dx = w(1/2) \qquad \forall\, w\, \in\, W_0 \tag{1.13}$$

where $W_0 = \{w\, \in\, W, w(0) = 0\}$."