Introduction to the Finite Element method

Gustavo C. Buscaglia

ICMC-USP, São Carlos, Brasil
gustavo.buscaglia@gmail.com

Motivation

- For **elliptic** and **parabolic** problems, the most popular approximation method is the FEM.
- It is **general**, not restricted to linear problems, or to isotropic problems, or to any subclass of mathematical problems.
- It is **geometrically flexible**, complex domains are quite easily treated, not requiring adaptations of the method itself.
- It is **easy to code**, and the coding is quite problem-independent. Boundary conditions are much easier to deal with than in other methods.
- It is **robust**, because in most cases the mathematical problem has an underlying variational structure (energy minimization, for example).

Overview

- **Galerkin approximations:** Differential, variational and extremal formulations of a simple 1D boundary value problem. Well-posedness of variational formulations. Functional setting. Strong and weak coercivity. Lax-Milgram lemma. Banach's open mapping theorem. Céa's best-approximation property. Convergence under weak coercivity. (2 lectures)
- **The spaces of FEM and their implementation:** (3 lectures)
- **Interpolation error and convergence:** (2 lectures)
- **Application to convection-diffusion-reaction problems:** (2 lectures)
- **Application to linear elasticity:** (2 lectures)
- **Mixed problems:** (2 lectures)
- **FEM for parabolic problems:** (2 lectures)

1 Galerkin approximations

1.1 Variational formulation of a simple 1D example

Let u be the solution of

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

The **differential formulation** (DF) of the problem requires $-u'' + u$ to be exactly equal to f in **all** points $x \in (0,1)$.
Multiplying the equation by any function v and integrating by parts (recall that

$$\int_0^1 w'\, z\; dx = w(1)z(1) - w(0)z(0) - \int_0^1 w\, z'\; dx \tag{1.2}$$

holds for all w and z that are *regular enough*) one obtains that u satisfies

$$\int_0^1 (u'\, v' + u\, v)\; dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 f\, v\; dx \qquad \forall\, v. \tag{1.3}$$

- The requirement "for all x" of the DF has become "for all functions v".
- Does equation (1.3) fully determine u?
- What happened with the boundary conditions?

Consider the following problem in **variational formulation** (VF): "Determine $u \in W$, such that $u(0) = u(1) = 0$ and that

$$\int_0^1 (u'\, v' + u\, v)\ dx = \int_0^1 f\, v\ dx \tag{1.4}$$

holds for all $v \in W$ satisfying $v(0) = v(1) = 0$."

Prop. 1.1 *The solution u of the DF (eq. 1.1) is also a solution of the VF if W consists of continuous functions of sufficient regularity. As a consequence, problem VF admits at least one solution whenever DF does.*

Proof. Following the steps that lead to the VF, it becomes clear that the only requirement for u to satisfy (1.4) is that the integration by parts formula (1.2) be valid. □

Exo. 1.1 *Show that the solution of*

$$\begin{cases} -u'' + u = f & \textit{in } (0,1) \\ u(0) = 0, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.5}$$

is a solution to: "Find $u \in W$ such that $u(0) = 0$ and that

$$\int_0^1 (u'\, v' + u\, v)\ dx = \int_0^1 f\, v\ dx \ + \ g\, v(1) \tag{1.6}$$

holds for all $v \in W$ satisfying $v(0) = 0$."

Consider the following problem in **extremal formulation** (EF): "Determine $u \in W$ such that it minimizes the function

$$J(w) = \int_0^1 \left(\frac{1}{2} w'(x)^2 + \frac{1}{2} w(x)^2 - f\, w \right) \, dx \tag{1.7}$$

over the functions $w \in W$ that satisfy $w(0) = w(1) = 0$."

Prop. 1.2 *The unique solution u of (1.1) is also a solution to EF. As a consequence, EF admits at least one solution.*

Proof. We need to show that $J(w) \geq J(u)$ for all $w \in W_0$, where

$$W_0 = \{ w \in W \,,\, w(0) = w(1) = 0 \}$$

Writing $w = u + \alpha v$ and replacing in (1.7) one obtains

$$J(u + \alpha\, v) = J(u) + \alpha \left[\int_0^1 (u'\, v' + u\, v - f\, v) \, dx \right] + \alpha^2 \int_0^1 \left(\frac{1}{2} v'(x)^2 + \frac{1}{2} v(x)^2 \right) \, dx$$

The last term is not negative and the second one is zero. □

Exo. 1.2 *Identify the EF of the previous exercise.*

Prop. 1.3 *Let u be the solution of*

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = 1, & u'(1) = g \in \mathbb{R} \end{cases} \tag{1.8}$$

then u is also a solution of "Determine $u \in W$ such that $u(0) = 1$ and that

$$\int_0^1 (u' \, v' + u \, v) \; dx = \int_0^1 f \, v \; dx \; + \; g \, v(1) \tag{1.9}$$

holds for all $v \in W$ satisfying $v(0) = 0$."
Further, defining for any $a \in \mathbb{R}$

$$W_a = \{w \in W, w(0) = a\},$$

u minimizes over W_1 the function

$$J(w) = \int_0^1 \left(\frac{1}{2} w'(x)^2 + \frac{1}{2} w(x)^2 - f \, w \right) \; dx \; - \; g \, w(1). \tag{1.10}$$

Exo. 1.3 *Prove the last proposition.*

Let us define the bilinear and linear forms corresponding to problem (1.1):

$$a(v,w) = \int_0^1 (v'w' + vw)\ dx \qquad\qquad \ell(v) = \int_0^1 f\,v\ dx \tag{1.11}$$

and the function $J(v) = \frac{1}{2}a(v,v) - \ell(v)$. Remember that W is a space of functions with some (yet unspecified) regularity and let $W_0 = \{w \in W,\, w(0) = w(1) = 0\}$.

The three formulations that we have presented up to now are, thus:

DF: Find a function u such that

$$-u''(x) + u(x) = f(x) \qquad \forall\, x \in (0,1), \qquad\qquad u(0) = u(1) = 0$$

VF: Find a function $u \in W_0$ such that

$$a(u,v) = \ell(v) \quad \forall\, v \in W_0$$

EF: Find a function $u \in W_0$ such that

$$J(u) \leq J(w) \qquad \forall\, w \in W_0$$

and **we know that the exact solution of DF is also a solution of VF and of EF**.

The logic of the construction is justified by the following

Theorem 1.4 *If W is taken as*

$$W = \{w : (0,1) \to \mathbb{R}, \int_0^1 w(x)^2 \, dx < +\infty, \int_0^1 w'(x)^2 \, dx < +\infty\} \stackrel{\text{def}}{=} H^1(0,1)$$

and if f is such that there exists $C \in \mathbb{R}$ for which

$$\int_0^1 f(x)\, w(x) \, dx \leq C \sqrt{\int_0^1 w'(x)^2 \, dx} \qquad \forall\, w \in W_0 \tag{1.12}$$

then problems (VF) and (EF) have one and only one solution, and their solutions coincide.

The proof will be given later, now let us consider its consequences:

- The differential equation has <u>at most one solution</u> in W.
- If the solution u to (VF)-(EF) is regular enough to be considered a solution to (DF), then u is <u>the solution</u> to (DF).
- If the solution u to (VF)-(EF) is <u>not</u> regular enough to be considered a solution to (DF), then (DF) <u>has no solution</u>.

$\Rightarrow$ (VF) is a <u>generalization</u> of (DF).

Exo. 1.4 *Show that* $W_0 \subset C^0(0,1)$. *Further, compute* $C \in \mathbb{R}$ *such that*

$$\max_{x\in[0,1]} |w(x)| \leq C\sqrt{\int_0^1 w'(x)^2\ dx} \qquad \forall\, w\, \in\, W_0$$

Hint: You may assume that $\int_0^1 f(x)\, g(x)\ dx \leq \sqrt{\int_0^1 f(x)^2\ dx}\sqrt{\int_0^1 g(x)^2\ dx}$ *for any* f *and* g *(Cauchy-Schwarz).*

Exo. 1.5 *Consider* $f(x) = |x - 1/2|^\gamma$. *For which exponents* γ *is* $\int_0^1 f(x)\, w(x)\ dx < +\infty$ *for all* $w\, \in\, W_0$*?*

Exo. 1.6 *Consider as* f *the "Dirac delta function" at* $x = 1/2$, *that we will denote by* $\delta_{1/2}$. *It can be considered as a "generalized" function defined by*

$$\int_0^1 \delta_{1/2}(x)\, w(x)\ dx = w(1/2) \qquad \forall\, w\, \in\, C^0(0,1)$$

Prove that $\delta_{1/2}$ *satisfies (1.12) and determine the analytical solution to (VF).*

Exo. 1.7 *Determine the DF and the EF corresponding to the following VF: "Find* $u\, \in\, W = H^1(0,1)$, $u(0) = 1$, *such that*

$$\int_0^1 (u'w' + uw)\ dx = w(1/2) \qquad \forall\, w\, \in\, W_0 \tag{1.13}$$

where $W_0 = \{w\, \in\, W, w(0) = 0\}$*."*

1.2 Variational formulations in general

Let V be a Hilbert space with norm $\|\cdot\|_V$. Let $a(\cdot,\cdot)$ and $\ell(\cdot)$ be bilinear and linear forms on V satisfying (continuity), for all $v,\, w \in V$,

$$a(v,w) \leq N_a \,\|v\|_V \,\|w\|_V, \qquad \qquad \ell(v) \leq N_\ell \,\|v\|_V \tag{1.14}$$

This last inequality means that $\ell \in V'$, the (topological) dual of V. The minimum N_ℓ that satisfies this inequality is called the norm of ℓ in V', i.e.

$$\|\ell\|_{V'} \overset{\text{def}}{=} \sup_{0\neq v\in V} \frac{\ell(v)}{\|v\|_V} \tag{1.15}$$

The abstract VF we consider here is:

$$\text{“Find } u \in V \text{ such that} \qquad a(u,v) = \ell(v) \qquad \qquad \forall\, v \in V\text{”} \tag{1.16}$$

Exo. 1.8 *Assume that V is finite dimensional, of dimension n, and let $\{\phi^1,\phi^2,\ldots,\phi^n\}$ be a basis. Show that (1.16) is then equivalent to*

$$\underline{V}^T \,\underline{\underline{A}}\,\underline{U} = \underline{V}^T \,\underline{L} \qquad \qquad \forall\,\underline{V} \in \mathbb{R}^n\ , \tag{1.17}$$

which in turn is equivalent to the linear system

$$\underline{\underline{A}}\ \underline{U} = \underline{L}\ ; \tag{1.18}$$

where

$$A_{ij} \overset{\text{def}}{=} a(\phi^j,\phi^i), \qquad L_i \overset{\text{def}}{=} \ell(\phi^i) \tag{1.19}$$

and $\underline{U}$ is the coefficient column vector of the expansion of u, i.e.,

$$u = \sum_{i=1}^{n} U_i\, \phi^i \tag{1.20}$$

Def. 1.5 *The bilinear form* $a(\cdot,\cdot)$ *is said to be* **strongly coercive** *if there exists* $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad\qquad \forall\, v \,\in\, V \tag{1.21}$$

Def. 1.6 *The bilinear form* $a(\cdot,\cdot)$ *is said to be* **weakly coercive** *(or to satisfy an* **inf-sup** *condition) if there exists* $\beta > 0$ *such that*

$$\sup_{0 \neq w \in V} \frac{a(v,w)}{\|w\|_V} \geq \beta \|v\|_V \qquad\qquad \forall\, v \,\in\, V \tag{1.22}$$

and

$$\sup_{0 \neq v \in V} \frac{a(v,w)}{\|v\|_V} \geq \beta \|w\|_V \qquad\qquad \forall\, w \,\in\, V \tag{1.23}$$

Exo. 1.9 *Prove that strong coercivity implies weak coercivity.*

Exo. 1.10 *Prove that, if* V *is finite dimensional, then* **(i)** $a(\cdot,\cdot)$ *is strongly coercive iff* $\underline{\underline{A}}$ *is positive definite* $(\underline{X}^T\ \underline{\underline{A}}\ \underline{X} > 0\ \forall\, \underline{X} \,\in\, \mathbb{R}^n)$*, and* **(ii)** $a(\cdot,\cdot)$ *is weakly coercive iff* $\underline{\underline{A}}$ *is invertible.*

Exo. 1.11 *Prove that, if* $a(\cdot,\cdot)$ *is weakly coercive, then the solution* u *of (1.16) depends continuously on the forcing* $\ell(\cdot)$*. Specifically, prove that*

$$\|u\|_V \leq \frac{1}{\beta}\ \|\ell\|_{V'} \tag{1.24}$$

Theorem 1.7 *Assuming V to be a Hilbert space, problem (1.16) is well posed for any $\ell \in V'$ if and only if* **(i)** *$a(\cdot,\cdot)$ is continuous, and* **(ii)** *$a(\cdot,\cdot)$ is weakly coercive.*

A simpler version of this result is known as **Lax-Milgram lemma**:

Theorem 1.8 *Assuming V to be a Hilbert space, if $a(\cdot,\cdot)$ is continuous and strongly coercive then problem (1.16) is well posed for any $\ell \in V'$.*

Proof. This proof uses the so-called "Galerkin method", which will be useful to introduce... the Galerkin method!
Let $\{\phi^i\}$ be a basis of V. Denoting $V_N = \mathrm{span}(\phi^1, \ldots, \phi^N)$ we can define $u_N \in V_N$ as the unique solution of $a(u_N, v) = \ell(v)$ for all $v \in V_N$. This generates a sequence $\{u_N\}_{N=1,2,\ldots}$ in V. Further, this sequence is bounded, because

$$\|u_N\|_V^2 \leq \frac{1}{\alpha}\, a(u_N, u_N) = \frac{1}{\alpha}\, \ell(u_N) \leq \frac{\|\ell\|_{V'}}{\alpha}\, \|u_N\|_V \quad \Rightarrow \quad \|u_N\|_V \leq \frac{\|\ell\|_{V'}}{\alpha}, \quad \forall\, N$$

Recalling the weak compactness of bounded sets in Hilbert spaces, there exists $u \in V$ such that a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$ for simplicity) converges to u weakly. It remains to prove that $a(u, v) = \ell(v)$ for all $v \in V$. To see this, notice that

$$a(u, \phi^i) = a(\lim_N u_N, \phi^i) = \lim_N a(u_N, \phi^i) = \ell(\phi^i)$$

where the last equality holds because $a(u_N, \phi^i) = \ell(\phi^i)$ whenever $N \geq i$. Uniqueness is left as an exercise. □

Exo. 1.12 *Prove uniqueness in the previous theorem (bounded sequences may have several accumulation points).*

Remark 1.9 *The space $L^2(a,b)$ (also denoted by $H^0(a,b)$) is the* Hilbert space *of functions $f:(a,b)\to\mathbb{R}$ such that $\int_a^b f^2(x)\,dx<+\infty$.*
The scalar product is

$$(f,g)_{L^2(a,b)} = \int_a^b f(x)g(x)\;dx \tag{1.25}$$

and accordingly

$$\|f\|_{L^2(a,b)} = (f,f)^{1/2}_{L^2(a,b)} = \sqrt{\int_a^b f^2(x)\;dx}\;. \tag{1.26}$$

Also of frequent use are the Hilbert spaces $H^1(a,b)$ and $H^2(a,b)$:

$$\begin{aligned}
H^1(a,b) &= \{f\in L^2(a,b)\mid f'\in L^2(a,b)\} && (1.27)\\
|f|_{H^1(a,b)} &= \|f'\|_{L^2(a,b)} && (1.28)\\
\|f\|_{H^1(a,b)} &= \|f\|_{L^2(a,b)}+|f|_{H^1(a,b)} && (1.29)\\
H^2(a,b) &= \{f\in H^1(a,b)\mid f''\in L^2(a,b)\} && (1.30)\\
|f|_{H^2(a,b)} &= \|f''\|_{L^2(a,b)} && (1.31)\\
\|f\|_{H^2(a,b)} &= \|f\|_{H^1(a,b)}+|f|_{H^2(a,b)} && (1.32)
\end{aligned}$$

Exo. 1.13 *Other* equivalent *norms can be defined in $H^1(a,b)$, e.g.,*

1. $|||f|||_{H^1(a,b)} = \left(\|f\|^2_{L^2(a,b)}+|f|^2_{H^1(a,b)}\right)^{1/2}$
2. $|||f|||_{H^1(a,b)} = \max\left(\|f\|_{L^2(a,b)},|f|_{H^1(a,b)}\right)$
3. $|||f|||_{H^1(a,b)} = \|f\|_{L^2(a,b)}+\|\ell\,f'\|_{L^2(a,b)}$, *where $\ell:(a,b)\to\mathbb{R}$ satisfies $0<\ell_{\min}\le\ell(x)\le\ell_{\max}$ for all $x\in(a,b)$. Notice that if $\ell(x)$ has dimensions of length then this norm is unit-consistent.*

Find the constants c and C such that $c\|f\| \leq |||f||| \leq C\|f\|$.

Remark 1.10 *For the spaces $H^1(a,b)$ and $H^2(a,b)$ to be complete, one needs a weaker definition of the derivative. For this purpose, one first introduces the space*

$$\mathcal{D}(a,b) = C_0^\infty(a,b) = \{\varphi \in C^\infty(a,b) \mid \varphi \text{ has compact support in } (a,b)\}\ . \tag{1.33}$$

Given a function $f : (a,b) \to \mathbb{R}$, if there exists $g : (a,b) \to \mathbb{R}$ such that

$$\int_a^b g(x)\,\varphi(x)\ dx = -\int_a^b f(x)\,\varphi'(x)\ dx\ , \qquad \forall\,\varphi \in \mathcal{D}(a,b)\ , \tag{1.34}$$

then we say that f' exists **in a weak sense**, *and that $f' = g$.*

Exo. 1.14 *Show that the function*

$$\phi(x) = \begin{cases} \exp\left(1/(|x|^2 - 1)\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{1.35}$$

belongs to $\mathcal{D}(\mathbb{R})$. By suitably shifting and scaling the argument of ϕ show that $\mathcal{D}(a,b)$ has infinite dimension for all $a < b$. (Hint: See Brenner-Scott, p. 27)

Exo. 1.15 *Consider $f(x) = 1 - |x|$ in the domain $(-1,1)$. Prove that its weak derivative is given by*

$$f'(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases}\ . \tag{1.36}$$

Prove also that f'' does not exist. (Hint: See Brenner-Scott, p. 28)

Exo. 1.16 *Let $f \in L^2(a,b)$, and let $V = H^1(a,b)$. Show that $\ell(v) = \int_a^b f(x)\,v(x)\,dx$ belongs to V' and that $\|\ell\|_{V'} \leq \|f\|_{L^2(a,b)}$.*

1.3 Galerkin approximations

The previous proof suggests a numerical method, the Galerkin method, to approximate the solution of a variational problem and thus of an elliptic PDE. The idea is simply to restrict the variational problem to a subspace of V that we will denote by V_h.

Discrete variational problem (Galerkin): Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\, v_h \in V_h \tag{1.37}$$

When the bilinear form $a(\cdot,\cdot)$ is symmetric and strongly coercive, this discrete probleme is equivalent to

Discrete extremal problem (Galerkin): Find $u_h \in V_h$ which minimizes over V_h the function

$$J(w) = \frac{1}{2}\, a(w,w) \;-\; \ell(w) \tag{1.38}$$

Exo. 1.17 *Prove this last assertion.*

The natural questions that arise are:

- Does u_h exist? Is it unique?
- Does u_h approximate u (the exact solution)?
- How difficult is it to compute u_h?

Does u_h exist? Is it unique?

Case 1) Strong coercivity of the form $a(\cdot,\cdot)$ over V

If $a(\cdot,\cdot)$ is strongly coercive over V, then

$$\inf_{0\neq w\in V} \frac{a(w,w)}{\|w\|_V^2} = \alpha > 0.$$

If $V_h \subset V$, then $a(\cdot,\cdot)$ is strongly coercive over V_h (because the infimum is taken over a smaller set). Then u_h exists and is unique as a consequence of Exo. 1.10.

Case 2) Weak coercivity of the form $a(\cdot,\cdot)$ over V

If $a(\cdot,\cdot)$ is just weakly coercive over V, then it may or may not be weakly coercive over V_h. Compare the two following conditions

$$(A)\ \inf_{w\in V}\sup_{v\in V}\frac{a(w,v)}{\|w\|_V\,\|v\|_V} = \beta > 0, \qquad\qquad (B)\ \inf_{w\in V_h}\sup_{v\in V_h}\frac{a(w,v)}{\|w\|_V\,\|v\|_V} = \beta_h > 0.$$

It is not true that (A)$\Rightarrow$(B) because the sup in (B) is taken over a smaller set. In this case the weak coercivity of the discrete problem must be proven independently, it is not inherited from the weak coercivity over the whole space V.

Does u_h approximate u?

Case 1) Strong coercivity of the form $a(\cdot,\cdot)$ over V

Lemma 1.11 (J. Céa) *If $a(\cdot,\cdot)$ and $\ell(\cdot)$ are continuous in V and $a(\cdot,\cdot)$ is strongly coercive, then*

$$\|u-u_h\|_V \leq \frac{N_a}{\alpha}\,\|u-v_h\|_V \qquad \forall\, v_h \in V_h \tag{1.39}$$

Proof. Notice the so-called **Galerkin orthogonality**:

$$a(u-u_h, v_h) = 0 \qquad \forall\, v_h \in V_h \tag{1.40}$$

which implies that $a(u-u_h, u-u_h) = a(u-u_h, u-v_h)$ for all $v_h \in V_h$. Using this,

$$\|u-u_h\|_V^2 \leq \frac{1}{\alpha} a(u-u_h, u-u_h) = \frac{1}{\alpha}\, a(u-u_h, u-v_h) \leq \frac{N_a}{\alpha}\,\|u-u_h\|_V\,\|u-v_h\|_V \qquad \forall v_h \in V_h$$

In other words, $\|u-u_h\|_V \leq C\,\inf_{v_h\in V_h}\|u-v_h\|_V$. □

Let h be a real parameter, typically a "mesh size". We say that a family $\{V_h\}_{h>0} \subset V$ satisfies the **approximability property** if:

$$\lim_{h\to 0} \operatorname{dist}(u, V_h) = \lim_{h\to 0}\ \inf_{v\,\in\,V_h}\ \|u-v\|_V \ = \ 0 \tag{1.41}$$

Corollary 1.12 *If $a(\cdot,\cdot)$ and $\ell(\cdot)$ are continuous in V, $a(\cdot,\cdot)$ is strongly coercive, and the family $\{V_h\}_{h>0} \subset V$ satisfies (1.41), then*

$$\lim_{h\to 0} u_h = u$$

in the sense of the norm $\|\cdot\|_V$.

If the strongly coercive bilinear form is **symmetric**, then $a(\cdot,\cdot)$ is a **scalar product** over V. In this case, Galerkin orthogonality corresponds to: **The Galerkin solution u_h is the orthogonal projection of u onto V_h.**
Further, the **energy norm** can be defined

$$\|v\|_a = \sqrt{a(v,v)}\ , \tag{1.42}$$

which satisfies the equivalence

$$\alpha^{\frac{1}{2}}\,\|v\|_V \ \leq\ \|v\|_a\ \leq\ N_a^{\frac{1}{2}}\,\|v\|_V\ . \tag{1.43}$$

Exo. 1.18 *Show that the Galerkin approximation is* **optimal** *in the energy norm,*

$$\|u-u_h\|_a \leq \|u-v_h\|_a\ , \qquad\qquad \forall\, v_h\ \in\ V_h\ , \tag{1.44}$$

without the constants that appear in Céa's lemma. Further show that the following sharper estimate holds:

$$\|u-u_h\|_V \leq \left(\frac{N_a}{\alpha}\right)^{\frac{1}{2}}\ \|u-v_h\|_V\ , \qquad\qquad \forall\, v_h\ \in\ V_h\ . \tag{1.45}$$

Case 2) Weak coercivity of the form $a(\cdot,\cdot)$ over V_h
Assume now that the weak coercivity constant β_h is positive for all $h > 0$, so that u_h exists and is unique. Notice that Galerkin orthogonality still holds.

Lemma 1.13 *If $a(\cdot,\cdot)$ and $\ell(\cdot)$ are continuous in V, and $a(\cdot,\cdot)$ is weakly coercive in V_h with constant $\beta_h > 0$, then*

$$\|u - u_h\|_V \leq \left(1 + \frac{N_a}{\beta_h}\right) \|u - v_h\|_V \qquad \forall\, v_h \in V_h \tag{1.46}$$

Proof. One begins by decomposing the error as follows (we omit the subindex V in the norm)

$$\|u - u_h\| \leq \|u - v_h\| + \|u_h - v_h\| \qquad \forall\, v_h \in V_h \tag{1.47}$$

and then using the weak coercivity

$$\|u_h - v_h\| \leq \frac{1}{\beta_h} \sup_{w_h \in V_h} \frac{a(u_h - v_h, w_h)}{\|w_h\|} = \frac{1}{\beta_h} \sup_{w_h \in V_h} \frac{a(u - v_h, w_h)}{\|w_h\|} \leq \frac{N_a}{\beta_h} \|u - v_h\|$$

Substituting this into (1.47) one proves the claim. □

Corollary 1.14 *Under the hypotheses of Lemma 1.13, if there exists $\beta_0 > 0$ such that $\beta_h > \beta_0$ for all h and the family $\{V_h\}_{h>0} \subset V$ satisfies (1.41), then*

$$\lim_{h \to 0} u_h = u$$

in the sense of the norm $\|\cdot\|_V$.

How difficult is it to compute u_h?

Let us go back to our problem $-u''+u = f$ in $(0,1)$ with $u(0) = u(1) = 0$, which in VF requires to compute $u \in H^1(0,1)$ satisfying the boundary conditions and such that

$$\int_0^1 [u'(x)\, v'(x) + u(x)\, v(x)]\ dx = \int_0^1 f(x)\, v(x)\ dx \tag{1.48}$$

Suitable spaces for the Galerkin approximation are, for example,

- $\mathcal{P}_k$: The polynomials of degree up to k.
- $\mathcal{F}_k$: The space generated by the functions $\phi^m(x) = \sin(m\,\pi\,x)$, $m = 1, 2, \ldots, k$.

Exo. 1.19 *Show that $a(\cdot,\cdot)$ is continuous and strongly coercive over $V = H^1(0,1)$ with the norm*

$$\|w\|_V \stackrel{def}{=} \left[\int_0^1 \left[w'(x)^2 + w(x)^2\right]\ dx\right]^{\frac{1}{2}}$$

Exo. 1.20 *Build a small program in Matlab or Octave (or something else) that solves the Galerkin approximation of problem (1.48) considering $f = \delta_{1/4}$ and the spaces $\mathcal{P}_k$ and/or $\mathcal{F}_k$, for some values of k. Compare the results to the analytical solution building plots of u and u_h. Also, build graphs of $\|u - u_h\|$ vs k.*

In general, however, the construction of spaces of global basis functions, as the ones above, is not practical because it leads to dense matrices. In the next chapter we will introduce the spaces of the FEM, which are characterized by having bases with small support and thus lead to sparse matrices.

Exercises

Reading assignment: Read Chapter 1 of Duran's notes (all of it).

Exo. 1.21 *Carry out the "easy computation" that shows that* $\underline{\underline{A}}$ *is the tridiagonal matrix such that the diagonal elements are* $2/h + 2h/3$ *and the extra-diagonal elements are* $-1/h + h/6$ *(Durán, page 3).*

Exo. 1.22 *Can a* <u>*symmetric*</u> *bilinear form be weakly coercive but not strongly coercive?*

Exo. 1.23 *To what variational formulation and what differential formulation corresponds the following extremal formulation?*
Find $u \in V$, V consisting of functions that are smooth in $(0, 1/2)$ and $(1/2, 1)$ but can exhibit a (bounded) discontinuity at $x = 1/2$, that minimizes the function

$$J(w) = \int_0^1 [w'(x)^2 + 2w(x)^2]\ dx + 4\ [w(1/2+) - w(1/2-)]^2 - \int_0^{1/2} 7\ w(x)\ dx - 9w(0) \tag{1.49}$$

where $w(1/2\pm)$ represent the values on each side of the discontinuity. Notice that the space V (is it a vector space really?) has no boundary condition imposed. What are the boundary conditions of the DF at $x = 0$ and $x = 1$?

Exo. 1.24 *Consider the bilinear form*

$$a(u, v) = \int_0^1 u'(x)\, v'(x)\, dx.$$

Prove that this form <u>*is not*</u> *strongly coercive in* $H^1(0, 1)$ *considering the norm*

$$\|w\|_{H^1} \stackrel{def}{=} \left\{ \int_0^1 \left[u'(x)^2 + u(x)^2\right]\ dx \right\}^{\frac{1}{2}}$$

and that <u>*it is*</u>*, with the same norm, in*

$$H_0^1(0, 1) \stackrel{def}{=} \{w \in H^1(0, 1)\, ,\, w(0) = w(1) = 0\}.$$

1.4 Variational formulations in 2D and 3D

The ideas are similar, but we need another integration by parts formula:

Lemma 1.15 *Let $f : \Omega \to \mathbb{R}$ be an integrable function, with Ω a Lipschitz bounded open set in $\mathbb{R}^d$ and $\partial_i f$ integrable over Ω, then*

$$\int_\Omega \partial_i f \; d\Omega = \int_{\partial\Omega} f \, n_i \; d\Gamma \tag{1.50}$$

Notice that this implies that

$$\int_\Omega \nabla \cdot \mathbf{v} \; d\Omega = \int_{\partial\Omega} \mathbf{v} \cdot \check{\mathbf{n}} \; d\Gamma \tag{1.51}$$

and that

$$\int_\Omega v \, \nabla^2 u \; d\Omega = \int_{\partial\Omega} v \, \nabla u \cdot \check{\mathbf{n}} \; d\Gamma - \int_\Omega \nabla v \cdot \nabla u \; d\Omega \tag{1.52}$$

We will also introduce the notation

Def. 1.16 *The* **Lebesgue space** $L^p(\Omega)$*, where $p \geq 1$, is the set of all functions such that their $L^p(\Omega)$-norm is finite,*

$$\|w\|_{L^p(\Omega)} \stackrel{def}{=} \left[\int_\Omega |w(x)|^p \; dx \right]^{\frac{1}{p}} \tag{1.53}$$

Exa. 1.17 (Poisson equation) *Consider the DF*

$$-\nabla^2 u = f \qquad in\ \Omega, \qquad u = 0 \qquad on\ \partial\Omega \tag{1.54}$$

where ∇ *is the gradient operator and* $\nabla^2 u = \sum_{i=1}^{d} \partial_{ii}^2 u$.

A suitable variational formulation is: Find $u \in V$ such that

$$a(u, v) = \ell(v) \qquad \forall\, v \in V$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; d\Omega, \qquad \ell(v) = \int_\Omega f\, v\, d\Omega \qquad \text{and} \tag{1.55}$$

$$V = H_0^1(\Omega) = \{w \in L^2(\Omega),\ \partial_i w \in L^2(\Omega)\, \forall i = 1, \ldots, d\,,\ w = 0 \text{ on } \partial\Omega\}$$

which is a Hilbert space with the norm

$$\|w\|_{H^1} = \left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right)^{\frac{1}{2}} \tag{1.56}$$

Exo. 1.25 *Prove that if* u *is a solution of the DF, then it solves the VF.*

Exo. 1.26 *Prove that* $a(\cdot,\cdot)$ *is continuous in* V*. Prove that* $\ell(\cdot)$ *is continuous in* V *if* $f \in L^2(\Omega)$*. Is this last condition necessary?*

Exo. 1.27 *Determine the EF of the Poisson problem.*

Exo. 1.28 *Is* $a(\cdot,\cdot)$ *strongly coercive?*

Exo. 1.29 *Let* Ω *be the unit circle. Determine for which exponents* γ *is the function* r^γ *in* $H^1(\Omega)$*.*

Exo. 1.30 *Assume that the domain* Ω *is divided into subdomains* Ω_1 *and* Ω_2 *by a smooth internal boundary* Γ*. Let* V *consist of functions such that their restrictions to* Ω_i *belong to* $H^1(\Omega_i)$ *and that are continuous across* Γ*. Determine the VF corresponding to the following EF:*Find $u \in V$ that minimizes

$$J(w) = \int_{\Omega_1} \frac{w^2 + \|\nabla w\|^2}{2}\, d\Omega + \int_{\Omega_2} \frac{3\|\nabla w\|^2}{2}\, d\Omega + \int_\Gamma (5\,w^2 - w)\, d\Gamma$$

over V.

Exo. 1.31 *Determine the DF that corresponds to the previous exercise.*