Introduction to the Finite Element method

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2 Finite Element Spaces

2.1 Introduction

A numerical method is a Galerkin finite element method if:

1. It is based on a variational formulation, i.e., Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\, v_h \in V_h \tag{2.1}$$

   where $a : V_h \times V_h \to \mathbb{R}$ is linear in its second argument and $\ell : V_h \to \mathbb{R}$ is a linear functional. Both, a and ℓ corresponds to the exact problem.

2. The discrete space V_h is a **finite element space.**

As an example, consider the 1D model problem previously introduced: "Determine $u_h \in V_h \subset H^1(0,1)$, such that $u_h(0) = 0$ and that

$$\int_0^1 (u_h'\, v_h' + \theta\, u_h\, v_h)\; dx = \int_0^1 f\, v_h\; dx \tag{2.2}$$

holds for all $v_h \in V_h$ satisfying $v_h(0) = 0$."

In the following sections the aim is to construct finite element spaces V_h to solve this problem. We begin with a few simple examples, introduce the concept of **degrees of freedom** and also some classical **finite element** basis.

2.2 1D examples

2.2.1 A space of polynomial functions in (a, b)

Consider the interval (a, b). We define the space as

$$V_h = P_k(a, b) = \{v, v = \sum_{i=0}^{k} \alpha_k \, x^k\} \tag{2.3}$$

e.g. $k = 1$

$$P_1(a, b) = \{v, v = \alpha + \beta \, x\} \tag{2.4}$$

This space has dimension 2. Once we have defined the space, we proceed like this:

1. Define a set of degrees of freedom $\{\sigma_1, \sigma_2\}$ (i.e., a set of linear functionals of P_k in $\mathbb{R}$).
2. Define $\{\phi^1(x), \phi^2(x)\}$ by the relation: $\sigma_i(\phi^j) = \delta_{ij}$
   (this is the Kronecker delta property, i.e., $\delta_{ij} = 1$ *if* $i = j$ *and* 0 *otherwise).*

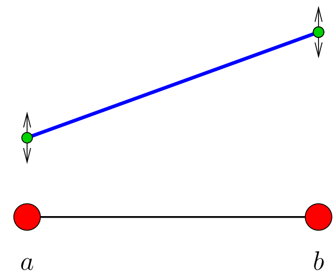
For instance, we can consider as degrees of freedom the value of the function at the end points of the interval:

$$\sigma_1(v) = v(a) \tag{2.5}$$

$$\sigma_2(v) = v(b) \tag{2.6}$$

To compute the basis, consider functions $\phi^j(x) = \alpha_j + \beta_j \, x$. In order to find α_j and β_j, $j = 1, 2$ we have two 2×2 systems to solve:

$$P_1(a,b)$$



$$\sigma_1(\phi^1) = \alpha_1 + \beta_1\, a = 1, \quad \sigma_1(\phi^2) = \alpha_2 + \beta_2\, a = 0$$
$$\sigma_2(\phi^1) = \alpha_1 + \beta_1\, b = 0, \quad \sigma_2(\phi^2) = \alpha_2 + \beta_2\, b = 1$$

Therefore, the basis is:

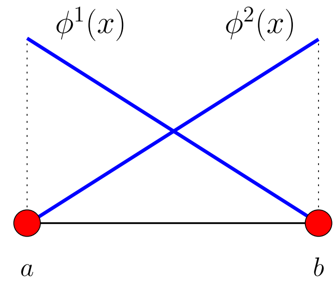
$$\phi^1(x) = \frac{b-x}{b-a}, \quad \phi^2(x) = \frac{x-a}{b-a} \tag{2.7}$$

The last choice seemed arbitrary, but it is a very **practical one**. If we want to describe a function in that space, the only thing I need is the value of the function at $x^1 = a$ and $x^2 = b$, since $\phi^i(x^j) = \delta_{ij}$, i.e.

$$v(x) = \sum_{i=1}^{2} U^j\, \phi^j(x) = \underbrace{U^1}_{v(a)}\, \phi^1(x) + \underbrace{U^2}_{v(b)}\, \phi^2(x) = v(a)\, \phi^1(x) + v(b)\, \phi^2(x)$$

As an exercise, compute the matrix $\underline{\underline{A}}$ for our model problem, i.e.

$$A_{ij} = a(\phi^i, \phi^j), \quad i,j = 1,2$$



We will write $\underline{\underline{A}}$ as sum of two matrices

$$\underline{\underline{A}} = \underline{\underline{K}} + \theta\, \underline{\underline{M}} \tag{2.8}$$

where

$$K_{ij} = a_d(\phi_i, \phi_j) = \int_a^b (\phi^i)'(\phi^j)'\,dx, \qquad M_{ij} = a_r(\phi_i, \phi_j) = \int_a^b \phi^i\,\phi^j\,dx$$

calculating the integrals

$$K_{11} = a_d(\phi^1, \phi^1) = \int_a^b (\phi^1)'(\phi^1)'\,dx = \frac{1}{b-a},$$

$$K_{12} = K_{21} = a_d(\phi^2, \phi^1) = \int_a^b (\phi^2)'(\phi^1)'\,dx = -\frac{1}{b-a}$$

$$K_{22} = a_d(\phi^2, \phi^2) = \int_a^b (\phi^2)'(\phi^2)'\,dx = \frac{1}{b-a}$$

and similarly we compute

$$M_{11} = a_r(\phi^1, \phi^1) = \int_a^b \phi^1\phi^1\,dx = \frac{b-a}{3}$$

$$M_{12} = M_{21} = a_r(\phi^1, \phi^2) = \int_0^1 \phi^1 \phi^2 \, dx = \frac{b-a}{6}$$

and so on ..., finally giving

$$\underline{\underline{A}} = \begin{bmatrix} \frac{1}{b-a} & -\frac{1}{b-a} \\ & \\ -\frac{1}{b-a} & \frac{1}{b-a} \end{bmatrix} + \theta \begin{bmatrix} \frac{b-a}{3} & \frac{b-a}{6} \\ & \\ \frac{b-a}{6} & \frac{b-a}{3} \end{bmatrix}$$

All these computations at individual intervals or elements will be useful later on when we construct aproximations on spaces defined on a collection of such elements.

Exo. 2.1 *Compute the basis when we choose as degrees of freedom:*

$$\sigma_1(v) = v\left(\frac{a+b}{2}\right) \tag{2.9}$$

$$\sigma_2(v) = v'\left(\frac{a+b}{2}\right) \tag{2.10}$$

Exo. 2.2 *Consider the space* $P_2(a,b) = \{v, v = \alpha_0 + \alpha_1 x + \alpha_2 x^2\}$. *Compute the basis* $\{\phi^1, \phi^2, \phi^3\}$ *when we choose as degrees of freedom*

$$\sigma_1(v) = v(a) \tag{2.11}$$

$$\sigma_2(v) = v\left(\frac{a+b}{2}\right) \tag{2.12}$$

$$\sigma_3(v) = v(b) \tag{2.13}$$

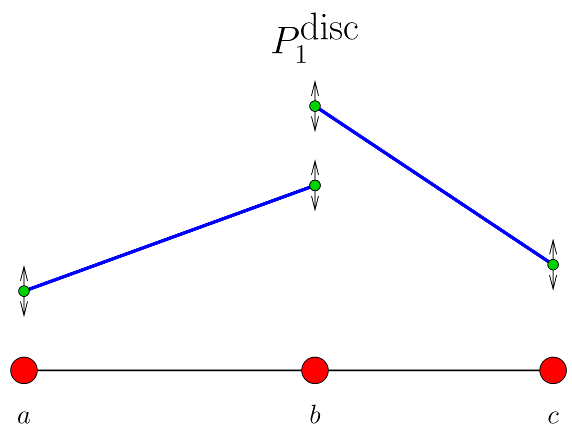
2.2.2 A polynomial space by parts

Consider the intervals (a, b) and (b, c) and define the space

$$V_h = P_k^{\text{disc}} = \{v,\ v|_{(a,b)} \in P_k(a, b),\quad v|_{(b,c)} \in P_k(b, c)\}$$

Consider again the case $k = 1$ for simplicity. The space has dimension 4. This is more or less evident if we notice that functions in this space are:

$$v = \begin{cases} \alpha + \beta\, x & \text{if } x \in (a, b) \\ \gamma + \epsilon\, x & \text{if } x \in (b, c) \end{cases}$$



Notice that such functions are not necessarily continuous at $x = b$ and therefore we have 4 degrees of freedom. For instance, choose for them:

$$
\begin{aligned}
\sigma_1(v) &= v(a) \\
\sigma_2(v) &= v(b^-) \\
\sigma_3(v) &= v(b^+) \\
\sigma_4(v) &= v(c)
\end{aligned}
$$

Now, we can compute the basis by using the relation $\sigma_i(\phi^j) = \delta_{ij}$. We write the basis function as

$$
\phi^j(x) = \begin{cases} \alpha_j + \beta_j\, x & \text{if } x \in (a,b) \\ \gamma_j + \epsilon_j\, x & \text{if } x \in (b,c) \end{cases}
$$

For $j = 1, \ldots, 4$ we have

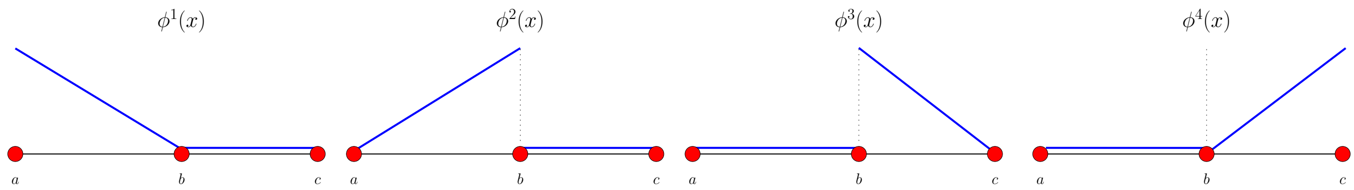
$$
\begin{aligned}
\sigma_1(\phi^j) &= \alpha_j + \beta_j\, a \;= \delta_{1j} \\
\sigma_2(\phi^j) &= \alpha_j + \beta_j\, b^- = \delta_{2j} \\
\sigma_3(\phi^j) &= \gamma_j + \;\epsilon_j\, b^+ = \delta_{3j} \\
\sigma_4(\phi^j) &= \gamma_j + \;\epsilon_j\, c \;\;= \delta_{4j}
\end{aligned}
$$

$$
\begin{cases} \alpha_1 + \beta_1\, a &= 1 \\ \alpha_1 + \beta_1\, b &= 0 \\ \gamma_1 + \epsilon_1\, b &= 0 \\ \gamma_1 + \epsilon_1\, c &= 0 \end{cases}, \quad
\begin{cases} \alpha_2 + \beta_2\, a &= 0 \\ \alpha_2 + \beta_2\, b &= 1 \\ \gamma_2 + \epsilon_2\, b &= 0 \\ \gamma_2 + \epsilon_2\, c &= 0 \end{cases}, \quad
\begin{cases} \alpha_3 + \beta_3\, a &= 0 \\ \alpha_3 + \beta_3\, b &= 0 \\ \gamma_3 + \epsilon_3\, b &= 1 \\ \gamma_3 + \epsilon_3\, c &= 0 \end{cases}, \quad
\begin{cases} \alpha_4 + \beta_4\, a &= 0 \\ \alpha_4 + \beta_4\, b &= 0 \\ \gamma_4 + \epsilon_4\, b &= 0 \\ \gamma_4 + \epsilon_4\, c &= 1 \end{cases},
$$

By inspection we find that the basis is:

$$\phi^1(x) = \begin{cases} \dfrac{b-x}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^2(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{if } x \in (b,c) \end{cases}$$

$$\phi^3(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \dfrac{c-x}{c-b} & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^4(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \dfrac{x-b}{c-b} & \text{if } x \in (b,c) \end{cases}$$



Now, if we define

$$V_1 = \{v, v|_{(a,b)} \in P_1(a,b),\ v(x) = 0\ \forall x \notin (a,b)\}$$
$$V_2 = \{v, v|_{(b,c)} \in P_1(b,c),\ v(x) = 0\ \forall x \notin (b,c)\}$$

which are the extensions by zero of the space P_1 we have defined at the beginning of the section, we can also define the space P_1^{disc} as their direct sum, i.e.

$$P_1^{\text{disc}} = V_1 \oplus V_2 = \{v,\ v = v_1 + v_2,\ v_i \in V_i\}$$

This already illustrates the importance of working on individual elements to further define more general spaces.

Now, let us try to find an approximation u_h of u from this space to solve our model problem when $\theta = 0$, but, first of all, recall the exact problem we are dealing with: "Determine $u \in V = H^1(0,1)$, such that $u(0) = 0$ and that

$$\int_0^1 u'\, v'\; dx = \int_0^1 f\, v\; dx$$

holds for all $v \in V$ satisfying $v(0) = 0$." In this case we are taking $a = 0,\ c = 1$. Notice that the integral above can be written as

$$\int_0^1 u'\, v'\; dx = \int_0^b u'\, v'\; dx + \int_b^1 u'\, v'\; dx$$

and similarly for the integral in the right hand side. Motivated by this, consider the following Galerkin formulation: "Determine $u_h \in V_h = P_1^{\text{disc}}$, such that $u_h(0) = 0$ and that

$$\int_0^b u_h'\, v_h'\; dx + \int_b^1 u_h'\, v_h'\; dx = \int_0^b f\, v_h\; dx + \int_b^1 f\, v_h\; dx$$

holds for all $v_h \in V_h$ satisfying $v_h(0) = 0$."

The discrete solution we are looking for is $u_h \in V_h$ and can be written as

$$u_h = \sum_{i=1}^{4} U_j \phi^j(x)$$

(i) First, we have to include the boundary condition in the definition of the space, for which we define

$$V_{h0} = \{v \in P_k^{\text{disc}}, v(0) = 0\}$$

Notice that this removes one degree of freedom, so this subspace has dimension 3 and it is spanned by $\{\phi^2, \phi^3, \phi^4\}$. This is like taking $U_1 = 0$ above.

(ii) Second, we have to compute the coefficients $a_d(\phi^i, \phi^j)$ of the matrix $\underline{\underline{K}} \in \mathbb{R}^{3\times 3}$ appearing in the linear system

$$\underline{\underline{K}}\,\underline{\mathrm{U}} = \underline{\mathrm{F}}$$

Considering the basis of V_{h0} to be the set of functions $\{\psi^1, \psi^2, \psi^3\} = \{\phi^2, \phi^3, \phi^4\}$, we compute the matrix:

$$\underline{\underline{K}} = \begin{bmatrix} a_d(\phi^2, \phi^2) & a_d(\phi^3, \phi^2) & a_d(\phi^4, \phi^2) \\ a_d(\phi^2, \phi^3) & a_d(\phi^3, \phi^3) & a_d(\phi^4, \phi^3) \\ a_d(\phi^2, \phi^4) & a_d(\phi^3, \phi^4) & a_d(\phi^4, \phi^4) \end{bmatrix} \tag{2.14}$$

and calculating the integrals we obtain ...

$$K_{11} = a_d(\phi^2, \phi^2) = \int_0^b (\phi^2)'(\phi^2)'\,dx + \int_b^1 (\phi^2)'(\phi^2)'\,dx = \int_0^b (\phi^2)'(\phi^2)'\,dx + 0 = \frac{1}{b},$$

$$K_{12} = K_{21} = a_d(\phi^3, \phi^2) = \int_0^b (\phi^3)'(\phi^2)'\,dx + \int_b^1 (\phi^3)'(\phi^2)'\,dx = \int_0^b 0\,(\phi^2)'\,dx + \int_b^1 (\phi^3)'\,0\,dx = 0$$

$$K_{13} = K_{31} = a_d(\phi^4, \phi^2) = \int_0^b (\phi^4)'(\phi^2)'\,dx + \int_b^1 (\phi^4)'(\phi^2)'\,dx = \int_0^b 0\,(\phi^2)'\,dx + \int_b^1 (\phi^4)'\,0\,dx = 0$$

... and so on, giving

$$\underline{\underline{K}} = \begin{bmatrix} \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{1-b} & -\frac{1}{1-b} \\ 0 & -\frac{1}{1-b} & \frac{1}{1-b} \end{bmatrix} \tag{2.15}$$

Notice that the term $K_{11} = a_d(\phi^2, \phi^2)$ is exactly what we had computed before when introducing the $P_1(a, b)$ space simply with $a = 0$, so, we could just have reused that result. Similarly for the second diagonal 2×2 block of (2.15),

$$\begin{bmatrix} \frac{1}{1-b} & -\frac{1}{1-b} \\ -\frac{1}{1-b} & \frac{1}{1-b} \end{bmatrix}$$

which is exactly the matrix we have computed before but in the interval (b, c) instead of (a, b) and taking $c = 1$.

Finally, notice also that $\underline{\underline{K}}$ **is singular!**

- Why did it fail?
- Is the space that we used a subset of $H^1(0, 1)$?

Answer: Functions in this space are discontinuous at $x = b$, therefore this space is not in $H^1(0, 1)$. Actually we have the following important theorem:

Theorem 2.1 *Let v be a* **piecewise-polynomial** *function on a partition of a domain Ω, then*

$$v \in H^1(\Omega) \iff v \in C^0(\bar{\Omega})$$

A more general version of the theorem as well as a proof for the 2D case and P_1 elements will be given later.

In the previous example, since functions in the space are discontinuous, their derivatives appearing in the integrals are Dirac delta functions at $x = b$, so, the integrals are not defined, however, since we partitioned the integrals, we naively proceed with the calculations and obtained a singular matrix. Following with this naive approach, it is interesting also to perform the computation of the system matrix when $\theta = 1$ and see what happens, for which it only remains the computation of matrix $\underline{\underline{M}}$

$$M_{ij} = \int_0^1 \phi^i \, \phi^j \, dx = \int_0^b \phi^i \, \phi^j \, dx + \int_b^1 \phi^i \, \phi^j \, dx$$

Again, we can reuse the results already obtained when describing the space P_1 for a single interval. The final matrix will be the sum of the previously computed $\underline{\underline{K}}$ and $\underline{\underline{M}}$.

$$\underline{\underline{A}} = \underline{\underline{K}} + \underline{\underline{M}} = \begin{bmatrix} \frac{1}{b} + \frac{b}{3} & 0 & 0 \\ 0 & \frac{1}{1-b} + \frac{1-b}{3} & -\frac{1}{1-b} + \frac{1-b}{6} \\ 0 & -\frac{1}{1-b} + \frac{1-b}{6} & \frac{1}{1-b} + \frac{1-b}{3} \end{bmatrix} \tag{2.16}$$

In this case, the matrix is not singular. For instance, if we take the function in the right hand side of the variational formulation to be the constant function $f = 1$ and we calculate the coefficients $\ell(\phi^i)$ of vector

$\underline{\text{F}} \in \mathbb{R}^3$ we get

$$F_1 = \ell(\phi^2) = \int_0^1 \phi^2 \, dx = \frac{b}{2}$$

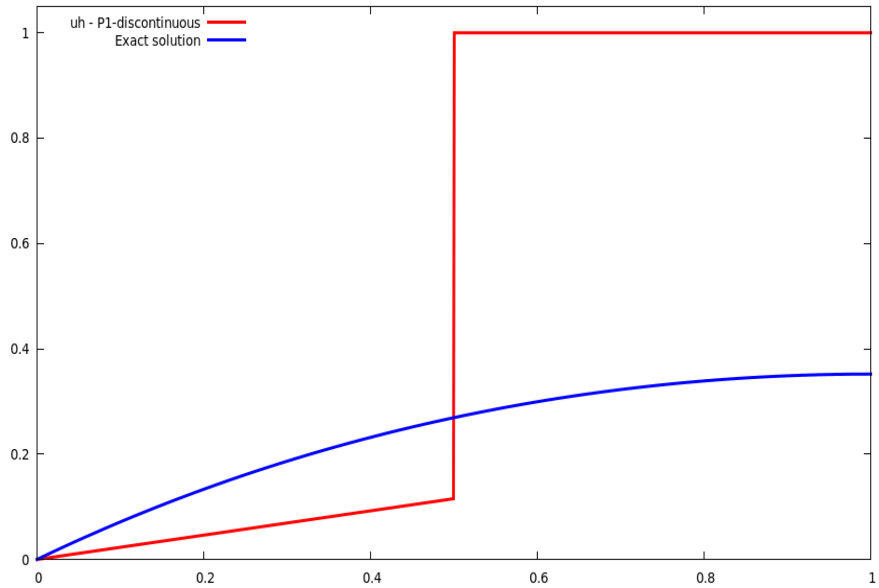
$$F_2 = \ell(\phi^3) = \int_0^1 \phi^3 \, dx = \frac{1-b}{2}$$

$$F_3 = \ell(\phi^4) = \int_0^1 \phi^4 \, dx = \frac{1-b}{2}$$

Taking now e.g. $b = 0.5$ and finally solving $\underline{\underline{A}}\,\underline{U} = \underline{F}$ we obtain $\underline{U} = \begin{bmatrix} 0.1154 & 1 & 1 \end{bmatrix}^T \Rightarrow u_h = 0.1154\,\phi^2(x) + \phi^3(x) + \phi^4(x)$, which is plotted below and compared with the exact solution for this problem.

This last example serves to illustrate that even when we are able to obtain some result, the approximation we are obtaining lacks of meaning as a consequence of an incorrect choice of the discrete space considered to solve the problem.

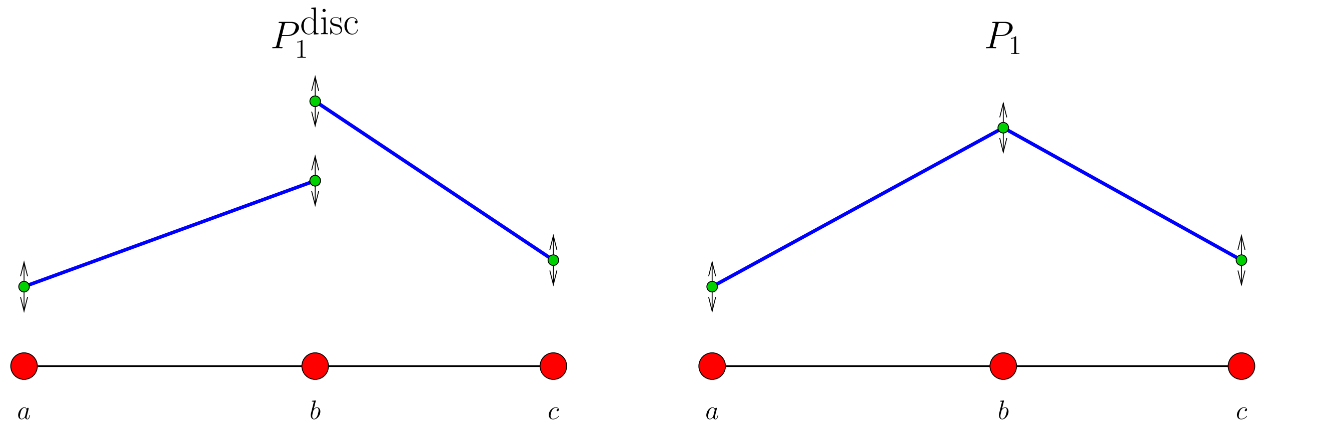
In the next section we remedy this by defining a space of continuous functions.



2.2.3 A P_1 continuous space

Consider the intervals (a, b) and (b, c). In the previous example "glue" the degree of freedom at $x = b$, of the interval to the left and to the right of this point, by imposing the restriction $v(b^-) = v(b^+)$. In this case we only have three degrees of freedom:

$$\begin{aligned} \sigma_1(v) &= v(a) \\ \sigma_2(v) &= v(b) \\ \sigma_4(v) &= v(c) \end{aligned}$$



therefore the space has dimension 3. This choice automatically leads to a space of continuous functions in (a, c) which we describe as

$$V_h = \{v,\ v|_{(a,b)} \in P_1(a,b),\ \ v|_{(b,c)} \in P_1(b,c)\}\ \cap\ C^0(a,c)$$

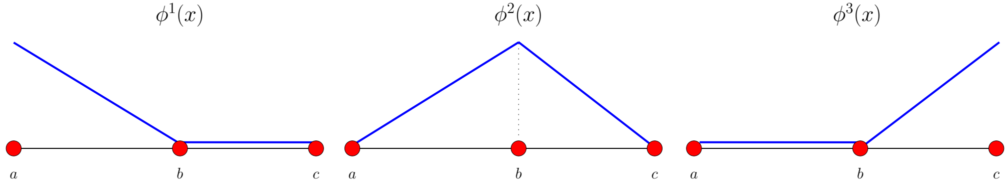
Again, considering

$$v(x) = \begin{cases} \alpha + \beta\, x & \text{if } x \in (a,b) \\ \gamma + \epsilon\, x & \text{if } x \in (b,c) \end{cases}$$

By inspection we find that the basis is:

$$\phi^1(x) = \begin{cases} \dfrac{b-x}{b-a} & \text{if } x \in (a,b) \\ \\ 0 & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^2(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in (a,b) \\ \\ \dfrac{c-x}{c-b} & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^3(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \\ \dfrac{x-b}{c-b} & \text{if } x \in (b,c) \end{cases}$$

which clearly satisfies that $\sigma_i(\phi^j) = \delta_{ij}$.



Exo. 2.3 *Compute the matrix $\underline{\underline{A}}$ for the model problem in this case and compare it with the one obtained when using the P_1^{disc} space. By computing the solution you will notice how good the approximation from this space is as illustrated below.*

Now, we generalize this to partitions of the interval with an increasing number of subintervals:

2.3 1D finite element meshes

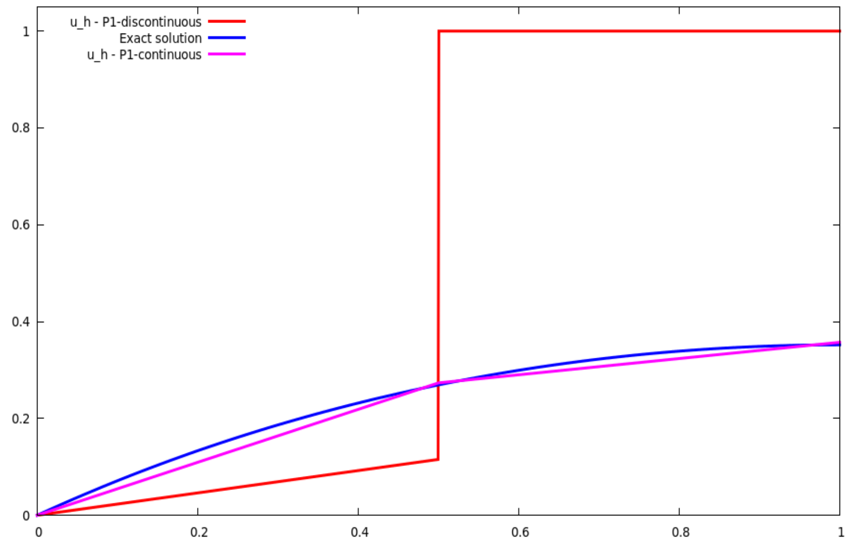
Let consider a partition $\mathcal{T}_h$ of $\Omega = [0, 1]$, i.e., an indexed collection of intervals

$$\bar{\Omega} = \bigcup_{j=1}^{N} I_j$$

where $I_j = [x^j, x^{j+1}]$ and the N_v **nodes** (arbitrarily numbered) are $0 = x^0 < x^1 < x^2 < \cdots < x^N < x^{N+1} = 1$. Define $h_i = x^{i+1} - x^i$ and

$$h = \max_j h_j$$

which is a measure of how fine the partition is.



2.3.1 A $P_1^{\text{disc}}(\mathcal{T}_h)$ (totally discontinuous) space in 1D

With the partition of Ω just defined, we begin by defining the spaces:

$$V_i = \{v, v|_{I_i} \in P_1(I_i),\ v(x) = 0\ \forall x \notin I_i\}$$

where $P_1(I_i) = P_1(x_i, x_{i+1})$ is the space P_1 for an individual interval the we introduced before.

Now, we define a totally discontinuous space associated to the partition $\mathcal{T}_h$ as the direct sum of these V_i's:

$$X_h(\mathcal{T}_h) = V_1 \oplus V_2 + \cdots \oplus V_N = \{v,\ v = v_1 + v_2 + \cdots + v_N,\ v_i \in V_i\}$$

This space has dimension equal to $N \times 2$, but it is not in $H^1(0,1)$.

Exo. 2.4 *Which degrees of freedom can be chosen in this case?*

2.3.2 $P_1(\mathcal{T}_h)$ conforming space in 1D

Now, if we "glue" the local degrees of freedom of the individual intervals at the corresponding common nodes of $\mathcal{T}_h$, which is equivalent to choosing as degrees of freedom **the values of the function at these nodes**, we naturally define a space of continuous functions

$$V_h = P_1(\mathcal{T}_h) = X_h(\mathcal{T}_h) \ \cap \ C^0(0,1)$$

and the basis functions will be

$$\phi^i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{if } x \in I_{i-1} \\[2ex] \dfrac{x_{i+1} - x}{h_i} & \text{if } x \in I_i \\[2ex] 0 & \text{otherwise} \end{cases}$$

- The dimension of V_h is equal to N_v;
- Since the degress of freedom are the values of the function at the nodes of $\mathcal{T}_h$ and the ϕ^i's are linearly independent, any function in V_h is uniquely determined precisely by these values, i.e.

$$v = \sum_{i=0}^{N+1} U^i \, \phi^i(x) = \sum_{i=0}^{N+1} v(x^i) \, \phi^i(x)$$

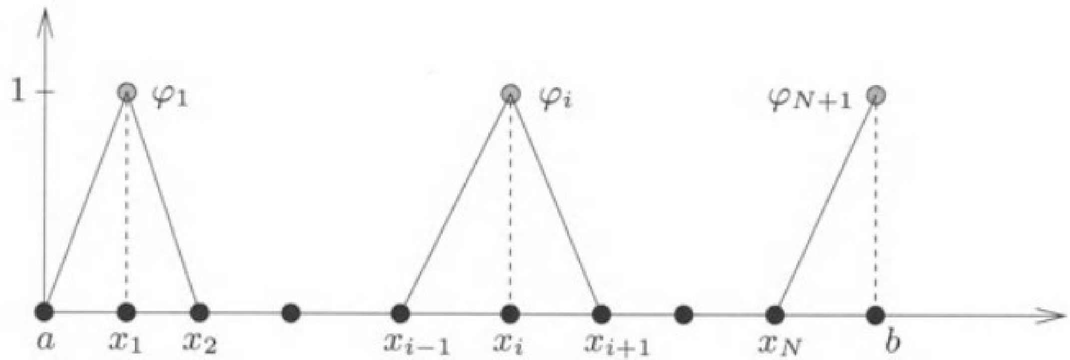
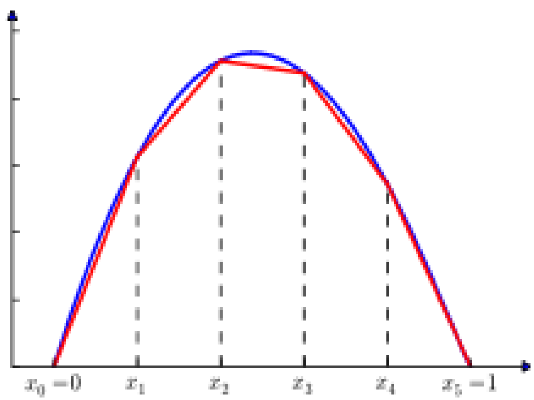


Fig. 1.1. One-dimensional hat functions.



- These functions are linear on each interval (or element) and continuous, but their derivatives are not defined in the classical sense at all points. Is $V_h \subset H^1(0,1)$?

  The answer is YES, as theorem 2.1 states.

We can use this space (introducing first the boundary conditions into its definition) to solve our model problem (see **Exo. 1.16**), and study $\|u - u_h\|_{H^1(0,1)}$ as we refine the partition. One would expect that the Galerkin approximation u_h from this space will converge to the solution u when $h \to 0$, which for this particular case is intuitive, because any continuous function can be approximated by polygonals with an increasing number of nodes.

We will study this in a more general setting in the following sections.

Exo. 2.5 *Do* **Exo. 1.16** *and read Duran's notes!*

2.4 2D examples

2.4.1 P_1 element for a triangle

Consider a triangle K in $\mathbb{R}^2$ with vertices $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$. We want to find a basis for

$$V_h = P_1(K) = \{v : K \to \mathbb{R},\ v = \alpha + \beta x + \gamma\, y\} \tag{2.17}$$

The space has dimension 3.

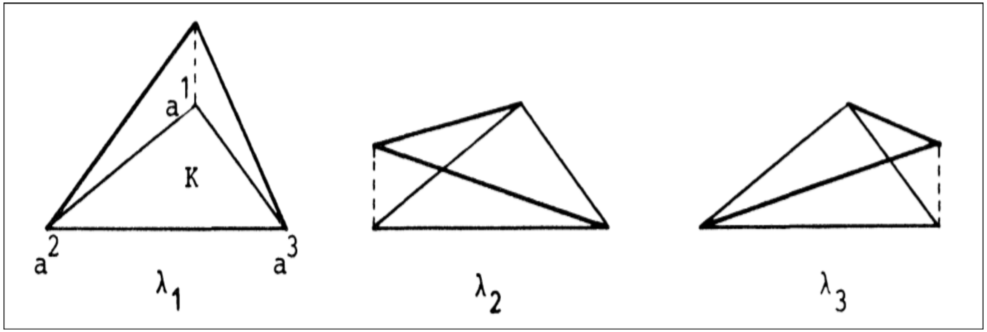
Again we start by defining the degrees of freedom. As done in previous examples we use the value of the function at a set of points, the vertices in this case

$$\sigma_i(v) = v(\mathbf{x}^i) \tag{2.18}$$

and the basis for $P_1(K)$ is defined by the relation $\sigma_i(\phi^j) = \delta_{ij}$. The coefficients of the basis functions are determined by solving the 3×3 systems:

$$\begin{array}{lll}
\alpha_1 + \beta_1\, x^1 + \gamma_1\, y^1 = 1, & \alpha_2 + \beta_2\, x^1 + \gamma_2\, y^1 = 0, & \alpha_3 + \beta_3\, x^1 + \gamma_3\, y^1 = 0 \\
\alpha_1 + \beta_1\, x^2 + \gamma_1\, y^2 = 0, & \alpha_2 + \beta_2\, x^2 + \gamma_2\, y^2 = 1, & \alpha_3 + \beta_3\, x^2 + \gamma_3\, y^2 = 0 \\
\alpha_1 + \beta_1\, x^3 + \gamma_1\, y^3 = 0, & \alpha_2 + \beta_2\, x^3 + \gamma_2\, y^3 = 0, & \alpha_3 + \beta_3\, x^3 + \gamma_3\, y^3 = 1
\end{array}$$

Notice that for simplicity of notation above we have used (x^i, y^i) instead of (x_1^i, x_2^i).



Exo. 2.6 *Write these functions for the master triangle $\hat{K}$ having as vertices* $((0,0),(1,0),(0,1))$.

Exo. 2.7 *Repeat the calculations for quadrilateral elements considering the bilinear functions* $v(x,y) = \alpha_0 + \alpha_1\, x + \alpha_2\, y + \alpha_3\, x\, y$ *and the value of the function at the vertices of the quadrangle as degrees of freedom.*

Exo. 2.8 *The values of the function at the vertices are not the only possible choice as degrees of freedom. Considering as degrees of freedom the line integrals*

$$\sigma_1(v) = \frac{1}{\|\mathbf{x}^2 - \mathbf{x}^3\|} \int_{\mathbf{x}^2}^{\mathbf{x}^3} v(s)\, ds \tag{2.19}$$

$$\sigma_2(v) = \frac{1}{\|\mathbf{x}^3 - \mathbf{x}^1\|} \int_{\mathbf{x}^3}^{\mathbf{x}^1} v(s)\, ds \tag{2.20}$$

$$\sigma_3(v) = \frac{1}{\|\mathbf{x}^1 - \mathbf{x}^2\|} \int_{\mathbf{x}^1}^{\mathbf{x}^2} v(s)\, ds \tag{2.21}$$

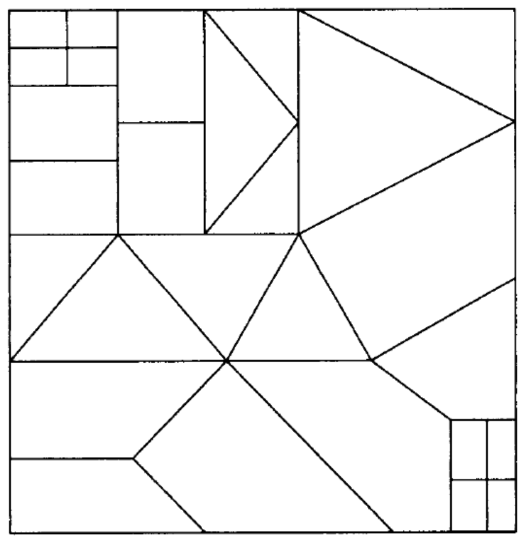
Calculate the basis for a $P_1(K)$*-triangle. This is called the Crouzeix-Raviart element.*

2.5 2D finite element meshes

Let consider a domain $\Omega \subset \mathbb{R}^2$ and for simplicity assume its boundary $\partial\Omega$ is a polygonal curve. Now, consider a partition $\mathcal{T}_h = \{K_i\}_{i=1}^N$ of Ω, such that

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{K}_i$$

where $K_i \cap K_j = \emptyset$ if $i \neq j$. $\mathcal{T}_h$ is called a triangulation of Ω.

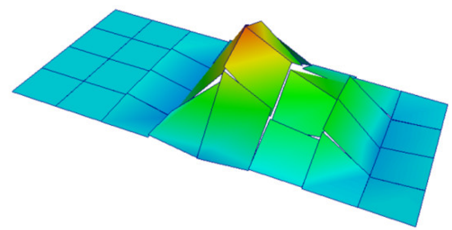


Given a triangulation like those shown, which types of spaces V_h can be constructed?

- We can construct spaces of **discontinuous functions**. If the partition has N_e triangular elements and we consider P_1-triangles for instance, we will have 3 (**local**) degrees of freedom per single triangle. Then a space of totally discontinuous functions associated to the partition $\mathcal{T}_h$ will be the direct sum of (local) P_1 spaces $V_K = \{v : K \to \mathbb{R}, v|_K \in P_1(K),\ v(\mathbf{x}) = 0\ \forall \mathbf{x} \notin K\},\quad K = 1, \ldots, N_e$, i.e.,

$$X_h(\mathcal{T}_h) = V_1 \oplus V_2 + \cdots \oplus V_N = \{v,\ v = v_1 + v_2 + \cdots + v_N,\ v_i \in V_i\}$$

  and its dimension will be $N_e \times 3$, but, remember that this space will not be in $H^1(\Omega)$ (see appendix).



- Also, we can construct spaces of **continuous functions**, but, it happens that this is not trivial in general for the so called nonconforming meshes, for which we have the following definition:

Def. 2.2 *A partition $\mathcal{T}_h$ of a domain Ω is* **conforming** *if $\bar{K}_i \cap \bar{K}_j$ is either*

- *empty, or,*
- *a vertex, or*

- *a complete edge.*

otherwise the partition is said to be **nonconforming**

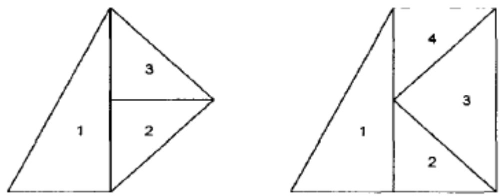


Figure 4.1. *Two examples of nonconforming triangulations. In both examples, the intersection of triangles 1 and 2 is a line segment that is not an edge of triangle 1.*

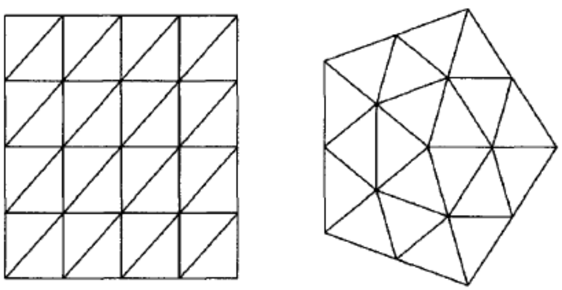


Figure 4.2. *Triangulations of two polygonal domains.*

2.5.1 $P_1(\mathcal{T}_h)$ conforming space in 2D

We proceed similarly to the 1D case. Given a **conforming triangulation** $\mathcal{T}_h$ of a polygonal domain we can build a space of continuous functions. Start with the space

$$X(\mathcal{T}_h) = \{v,\ v|_{K_i} \in P_1(K_i)\ \forall K_i \in \mathcal{T}_h\} \tag{2.22}$$

where $v|_K$ denotes the restriction of v to K and $P_1(K)$ is the space of polynomial functions of degree ≤ 1 on triangle K that we have already defined in subsection 2.4.1

We define as degrees of freedom the value of the function at the nodes of the triangulation.

Since we are assuming now that $\mathcal{T}_h$ is conforming, each vertex of any triangle can only be a vertex of other triangles an cannot be on an edge. Thus, we can "glue" the (local) degrees of freedom of the individual triangles. This naturally leads to the following description of the space we have constructed

$$V_h = X(\mathcal{T}_h)\ \cap\ C^0(\bar{\Omega}) = \{v \in C^0(\bar{\Omega}),\ v|_K \in P_1(K)\ \ \forall K \in \mathcal{T}_h\}$$

We can construct a basis for this space immediately. Let us assume $\mathcal{T}_h$ has N_v vertices whose coordinates are $\{\mathbf{x}^i\}_{i=1}^{N_v}$. Let $\phi^i,\ i = 1, \ldots, N_v$ be the functions that satisfies

$$\phi^i(\mathbf{x}^j) = \delta_{ij} \tag{2.23}$$

whose restriction to element K having j as one of its vertices is the corresponding function in $P_1(K)$ and 0 otherwise. Any function $v = \sum_{i=1}^{N_v} v(\mathbf{x}^i)\,\phi^i(x) \in V_h$ is uniquely determined by the degrees of freedom that are precisely the values of the function at the N_v nodes of $\mathcal{T}_h$. Notice that

- $\{\phi^j\}_{j=1}^{N_v}$ are linearly independent;

- $V_h = \text{span}\{\phi^j\} = \{v_h, v_h = \sum_{j=1}^{N_v} a^j \phi^j\}$;
- $\dim(V_h) = N_v$.

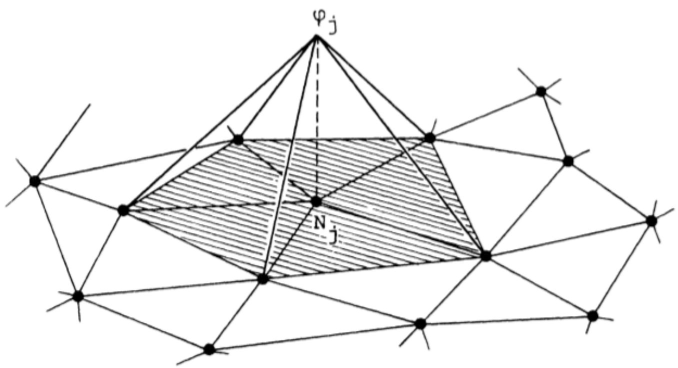
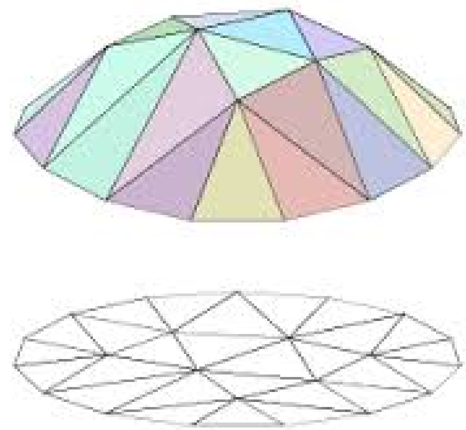


Fig 1.9 The basis function φ_j.



Exo. 2.9 *Show that functions of V_h are actually continuous at the common edge between two triangles of $\mathcal{T}_h$.*

Exo. 2.10 *Noticing that the support of function ϕ^j are all the elements sharing node j, what are the consequences for the matrix $\underline{\underline{A}}$ ($A_{ij} = a(\phi^i, \phi^j)$), when choosing such space to compute an approximation to u?*

2.6 More examples of finite elements and their associated global spaces

2.6.1 P_2 triangular element

Consider a triangle K in $\mathbb{R}^2$ with vertices $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$. We want to find a basis for

$$V_h = P_2(K) = \{v : K \to \mathbb{R},\ v = \alpha_0 + \alpha_1 x + \alpha_2\, y + \alpha_3\, x^2 + \alpha_4 y^2 + \alpha_5\, x\, y\}$$

The space has dimension 6 since an element of $P_2(K)$ is determined by six independent parameters. Any function is uniquely determined by its values at:

- the vertices of the triangle;
- the midpoints of the three edges.

Take two points $\mathbf{x}^i$ and $\mathbf{x}^j$. If a function v belongs to $P_2(K)$ then

$$v\left((1-s)\,\mathbf{x}^i + s\,\mathbf{x}^j\right) \in P_2(s) = \{w,\ w = \beta_0 + \beta_1\, s + \beta_2\, s^2\}$$

where $0 \leq s \leq 1$, this is, the function restricted to the straight segment joining $\mathbf{x}^i$ and $\mathbf{x}^j$ of the triangle, is a parabolic function, which is uniquely determined by its values at the three points.

When considering the master triangle $\hat{K}$ used above we have:

$$\hat{\psi}^1 = (1-\hat{x}-\hat{y})\,(1-2\hat{x}-2\hat{y}), \quad \hat{\psi}^2 = \hat{x}\,(2\hat{x}-1), \quad \hat{\psi}^3 = \hat{y}\,(2\hat{y}-1)$$
$$\hat{\psi}^4 = 4\,\hat{x}\,\hat{y}, \quad \hat{\psi}^5 = 4\,\hat{y}\,(1-\hat{x}-\hat{y}), \quad \hat{\psi}^6 = 4\,\hat{x}\,(1-\hat{x}-\hat{y}) \tag{2.24}$$

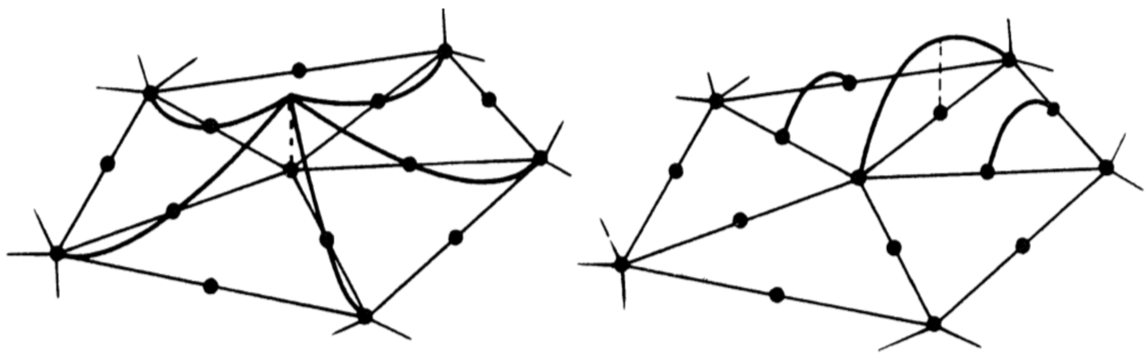
These functions clearly satisfy $\phi^i(\vec{p}^j) = \delta_{ij}$, where $\vec{p}^j$ corresponds to the vertices for $j = 1, 2, 3$ and to the midpoints of sides for $j = 4, 5, 6$.

Exo. 2.11 *How the vertices are numbered in this master triangle?*

Given a conforming triangulation $\mathcal{T}_h$, we want to construct a space of continuous functions as we did before, i.e., "gluing" together the degrees of freedom of all the $P_2(K)$-triangles in $\mathcal{T}_h$, that share a vertex or a midpoint. In order to do so, simply fix the value of the function at all vertices and at all midpoints (on edges shared by two triangles). The resulting function will be **continuous** and belong to the space:

$$V_h = P_2(\mathcal{T}_h) = \{v \in C^0(\bar{\Omega}),\ v|_K \in P_2(K)\ \ \forall K \in \mathcal{T}_h\}$$

Exo. 2.12 *Which is the dimension of V_h?*



2.6.2 Triangular elements of arbitrary degree

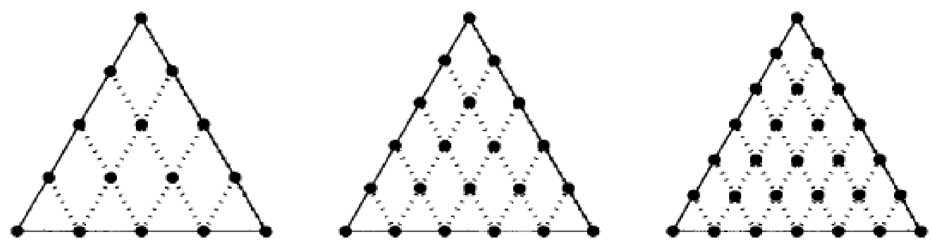
We can generalize the finite element spaces we have considered to construct space of continuous piecewise polynomial functions of arbitrary degree k on a triangle. The placement of the nodes on the triangle is determined by:

(i) On each edge there must by $k+1$ nodes since a one dimensional polynomial of degree k has $k+1$ degrees of freedom. Each edge has two vertices and the other $k-1$ nodes will be regularly spaces between them;

(ii) A polynomial of degree k in two variables is determined by

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

parameters. Therefore, the number of interior nodes will be $\frac{(k+1)(k+2)}{2}-3k$.



When using these Lagrange triangles the stiffness matrix $\underline{\underline{K}}$ ($K_{ij}=a_d(\phi^i,\phi^j)$) may become ill conditioned as the finite element mesh is refined and is a consequence of the basis choosen. This problem can be circunvented by choosing other basis functions.

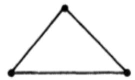
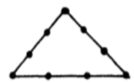
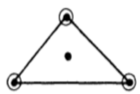
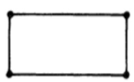
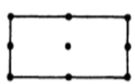
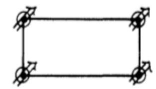
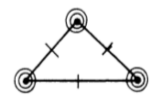
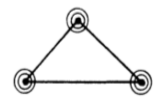
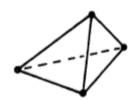
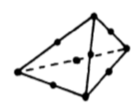
Given a triangulation of a domain Ω it is interesting to know the relation between the number of vertices N_v, the number of edges N_{edges}, the number of elements N_e. This is given by the Euler relations:

Lemma 2.3 Euler relations.
Let $\mathcal{T}_h$ be a conforming partition of a polygonal domain $\Omega \subset \mathbb{R}^2$, then

$$\begin{aligned} N_e - N_{edges} + N_v &= 1 - I \\ N_v^\partial - N_{edges}^\partial &= 0 \end{aligned}$$

where I is the number of holes in Ω. In particular, if the elements are polygons with ν vertices

$$2N_{edges} + N_{edges}^\partial = \nu\, N_e$$

Degrees of freedom Σ Geometry	Function space P_K	Degree of continuity of corresponding FEM-space V_h
 3	$P_1(K)$	C^0
 6	$P_2(K)$	C^0
 10	$P_3(K)$	C^0
 10	$P_3(K)$	C^0
 4	$Q_1(K)$	C^0
 9	$Q_2(K)$	C^0
 16	$Q_3(K)$	C^1
 2	$P_1(K)$	C^0
 3	$P_2(K)$	C^0
 4	$P_3(K)$	C^1
 21	$P_5(K)$	C^1
 18	$P_5'(K)$ (see Problem 3.7)	C^1
 4	$P_1(K)$	C^0
 10	$P_2(K)$ (See Problem 3.4)	C^0

2.7 General definition of a Finite element

Def. 2.4 *(Ciarlet) A finite element in $\mathbb{R}^d$ (typically $d = 1$, 2 or 3) is a triplet (K, P_K, Σ_K) where*

(i) K is a closed bounded subset of $\mathbb{R}^d$ with a non-empty interior and Lipschitz boundary;

(ii) P_K is a finite dimensional space of functions defined over K of dimension n;

(iii) Σ_K is a set of n linear functionals $\{\sigma_i\}_{i=1,\ldots,n}$ such that for any real scalars α_i, $i = 1, \ldots, n$ there exist an unique function $p \in P_K$ that satisfies

$$\sigma_i(p) = \alpha_i \tag{2.25}$$

We say that Σ_K is P_K-unisolvent.

Def. 2.5 *The linear forms $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ that are a basis for P'_K are called the degrees of freedom.*

Exo. 2.13 *Show that (iii) is equivalent to:*

$$\sigma_i(p) = 0 \Leftrightarrow p = 0, \quad i = 1, \ldots, n \tag{2.26}$$

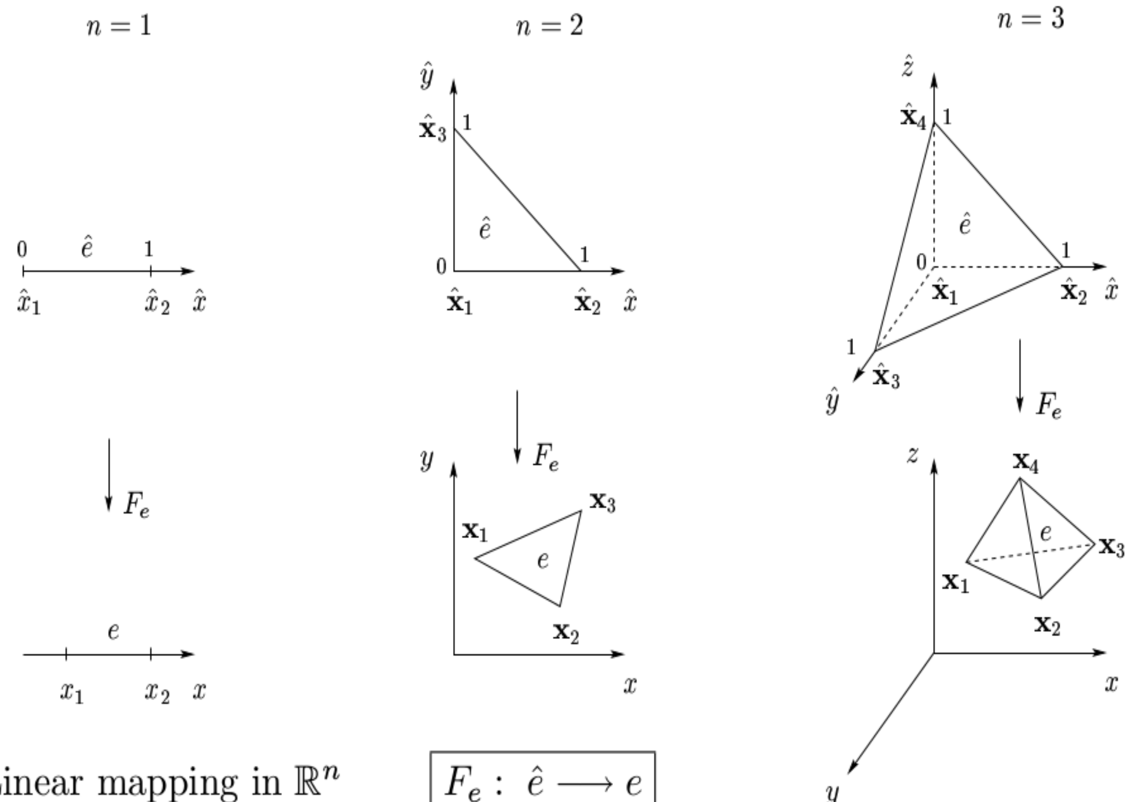
Prop. 2.6 *There exists a basis $\{\psi^1, \psi^2, \ldots, \psi^n\}$ in P_K such that*

$$\sigma_i(\psi^j) = \delta_{ij}, \quad 1 \leq i, j \leq n \tag{2.27}$$

Def. 2.7 $\{\psi^1, \psi^2, \dots, \psi^n\}$ *are called the basis functions.*

Def. 2.8 *Let* $\{K, P_K, \Sigma\}$ *be a finite element. If there is a set of points* $\{\vec{a}^1, \dots, \vec{a}^n\}$ *in* K *such that for all* $p \in P_K$, $\sigma_i(p) = p(\vec{a}^i)$ $i = 1, \dots, n$, $\{K, P_K, \Sigma\}$ *is called a* **Lagrange finite element**.

2.8 Affine family of finite elements



Linear mapping in $\mathbb{R}^n$

$$F_e : \hat{e} \longrightarrow e$$

The concept of affine family of finite elements is important because:

- The computation of the coefficients of the linear systems is done on a reference finite element.
- For such affine families, the interpolation theory that is the basis of most convergence theorems is easier to develop.

Def. 2.9 *A family of finite elements is called an affine family if all its elements are affine equivalent to a single reference or* **master** *element.*

An affine transformation $F_K : \hat{K} \to K$ of the reference element $\hat{K}$ with vertices $\hat{\mathbf{x}}^i$ onto an element K with vertices $\mathbf{x}^i$ is defined by:

$$F_K(\hat{\mathbf{x}}) = B_K \cdot \hat{\mathbf{x}} + \mathbf{b}_K, \quad B_K \in \mathbb{R}^{d\times d}, \; \mathbf{b} \in \mathbb{R}^d \tag{2.28}$$

We have

$$d = 1, \qquad B_K = x^2 - x^1, \qquad b_K = x^1$$

$$d = 2, \qquad B_K = \begin{bmatrix} x^2 - x^1 & x^3 - x^1 \\ y^2 - y^1 & y^3 - y^1 \end{bmatrix}, \qquad b_K = \begin{bmatrix} x^1 \\ y^1 \end{bmatrix}$$

Note that in this case we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^1 \\ y^1 \end{bmatrix} (1 - \hat{x} - \hat{y}) + \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \hat{x} + \begin{bmatrix} x^3 \\ y^3 \end{bmatrix} \hat{y} = \sum_{j=1}^{3} \mathbf{x}^j \, \hat{\psi}^j(\hat{\mathbf{x}})$$

Exo. 2.14 *Write the affine mapping for a tetrahedral element (see figure in previous slide to see the definition of the master or reference element).*

2.8.1 Properties of the affine mapping

(a) Vertices are mapped onto vertices:

$$\mathbf{x}^i = F_K(\hat{\mathbf{x}}^i)$$

(b) Midpoints of sides are mapped onto midpoints of sides:

$$\mathbf{x}^{ij} = \frac{\mathbf{x}^i + \mathbf{x}^j}{2} = F_K\left(\frac{\hat{\mathbf{x}}^i + \hat{\mathbf{x}}^j}{2}\right) = F_K(\hat{\mathbf{x}}^{ij})$$

(c) Barycenters are mapped onto barycenters

$$\mathbf{x}^{ijk} = \frac{\mathbf{x}^i + \mathbf{x}^j + \mathbf{x}^k}{3} = F_K\left(\frac{\hat{\mathbf{x}}^i + \hat{\mathbf{x}}^j + \hat{\mathbf{x}}^k}{3}\right) = F_K(\hat{\mathbf{x}}^{ijk})$$

(d) For a function ψ defined on K, we define $\hat{\psi}$ on $\hat{K}$ by

$$\hat{\psi}(\hat{\mathbf{x}}) = \psi\left(F_K(\hat{\mathbf{x}})\right) = \psi(\mathbf{x})$$

Therefore, if function ψ is a polynomial of degree k on K, $\hat{\psi}$ is also a polynomial of degree k on $\hat{K}$.

(e) The derivatives of ψ and $\hat{\psi}$ are related by

$$\nabla\psi(\mathbf{x}) = B_K^{-T} \cdot \hat{\nabla}\hat{\psi}(\hat{\mathbf{x}}) \tag{2.29}$$

(f) $|\det B_K| = \frac{\text{meas}(K)}{\text{meas}(\hat{K})}$

Exo. 2.15 *Show all the previous properties.*

Proof. of property (e). First note that:

$$\hat{\mathbf{x}} = B_K^{-1} \cdot \mathbf{x} + \tilde{\mathbf{b}}_K$$

where $\tilde{\mathbf{b}}_K = -B_K^{-1} \cdot \mathbf{b}_K$. Using index notation

$$\hat{x}_k = \left[B_K^{-1}\right]_{k\ell} x_\ell + \left[\tilde{\mathbf{b}}_K\right]_k \quad \Rightarrow \quad \frac{\partial \hat{x}_k}{\partial x_\ell} = \left[B_K^{-1}\right]_{k\ell}$$

Now, applying the chain rule

$$\frac{\partial \psi}{\partial x_k}(\mathbf{x}) = \frac{\partial \hat{\psi}}{\partial \hat{x}_\ell}\left(F_K^{-1}(\mathbf{x})\right) \frac{\partial \hat{x}_\ell}{\partial x_k} = \frac{\partial \hat{\psi}}{\partial \hat{x}_\ell}\left(F_K^{-1}(\mathbf{x})\right) \left[B_K^{-1}\right]_{\ell k} = \left[B_K^{-T}\right]_{k\ell} \frac{\partial \hat{\psi}}{\partial \hat{x}_\ell}\left(F_K^{-1}(\mathbf{x})\right)$$

or

$$\nabla \psi(\mathbf{x}) = B_K^{-T} \cdot \hat{\nabla} \hat{\psi}\left(F_K^{-1}(\mathbf{x})\right)$$

The case of P_1 linear elements is particularly simple

$$\hat{\psi}^1 = 1 - \hat{x} - \hat{y}, \quad \hat{\psi}^2 = \hat{x}, \quad \hat{\psi}^3 = \hat{y}$$

whose gradients are:

$$\hat{\nabla}\hat{\psi}^1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \hat{\nabla}\hat{\psi}^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{\nabla}\hat{\psi}^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so, the only thing we have to do in order to compute $\nabla \psi^j$, $j = 1, 2, 3$, is trasposing the inverse of B_K and multlying by those constant vectors.

□

Prop. 2.10 *If K and $\hat{K}$ are affine equivalent and if the triplet $(\hat{K}, \hat{P}, \hat{\Sigma}_K)$ is a finite element, then we can define (K, P_K, Σ) and it is a finite element.*

Proof. Let $F_K : \hat{K} \to K$ be the affine mapping. We have to show how to contruct P_K and Σ based on $\hat{P}$ and $\hat{\Sigma}$. We define for any $\hat{v} \in \hat{P}$ the function $v \in P_K$ by $v(\mathbf{x}) = \hat{v}\left(F_K^{-1}(\mathbf{x})\right)$

$$P_K = \{v : K \to \mathbb{R},\ \hat{v} \in \hat{P}\} \tag{2.30}$$

and

$$\Sigma_K = \{\sigma : P_K \to \mathbb{R}, \sigma(v) = \hat{\sigma}(\hat{v})\ \forall\ \hat{v} \in \hat{P} \text{ and } \hat{\sigma} \in \hat{\Sigma}\} \tag{2.31}$$

□

Two comments are in order here:

- As we said before, computation of the coefficients of $\underline{\underline{A}}$ when solving discrete variational problems is easily done when working in the master element $\hat{K}$. We rarely compute the basis functions on the real element K. All the information we need regarding the element geometry is in the affine mapping.

- When constructing the degrees of freedom in this way, the only ones that are preserved when passing from one element to the other are the degrees of freedom involving the values of the function at a set of points, i.e., the **Lagrangian** degrees of freedom. The case of the so called **Hermitian** elements, involving the derivatives of the function at a set of points as degrees of freedom, have to be considered differently.

2.9 Practical aspects

Now, we discuss practical aspects and introduce some "technology" needed for the actual computation of matrix $\underline{\underline{A}}$ considering some of the finite element spaces constructed. **Here, we will be using affine families of finite elements.**

Computation of matrix $\underline{\underline{A}}$ tyically involves integrals of the type

$$A_{ij} = a(\phi^i, \phi^j) = \int_\Omega \left[\phi^i(\mathbf{x})\,\phi^j(\mathbf{x}) + \nabla\phi^i(\mathbf{x})\cdot\nabla\phi^j(\mathbf{x})\right] d\Omega$$

So, consider now a finite element partition $\mathcal{T}_h$ of Ω, i.e.

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{K}_i$$

Let us consider the first term in the integral above. We can compute then the integral summing over all the elements

$$M_{ij} = \int_\Omega \phi^i(\mathbf{x})\,\phi^j(\mathbf{x})\,d\Omega = \sum_{K_m\in\mathcal{T}_h} \int_{K_m} \phi^i(\mathbf{x})|_{K_m}\,\phi^j(\mathbf{x})|_{K_m}\,dK \tag{2.32}$$

The notation above is redundant, because we are integrating on K_m. Now, we make use of the affine mapping we have previously introduced. The idea is to transform the integral over K_m into an integral over $\hat{K}$ which is **easier** to handle. By doing the change of variables

$$\int_{K_m} \phi^i(\mathbf{x})|_{K_m}\,\phi^j(\mathbf{x})|_{K_m}\,dK = \int_{\hat{K}} \phi^i(F_K(\hat{\mathbf{x}}))\,\phi^j(F_K(\hat{\mathbf{x}}))\,|J_{K_m}|\,d\hat{K}$$

where

$$|J_{K_m}| = |\mathrm{det}B_{K_m}|$$

i.e., the determinant of the Jacobian of the affine transformation for element K.

The idea is to use the basis functions defined on the master element and not the functions defined on the real element. As an example, consider the case of a triangular mesh $\mathcal{T}_h$ and P_1 linear elements. We have constructed basically two types of spaces (see figure below):

- *Space of totally discontinous functions*

$$X_h(\mathcal{T}_h) = P_1^{\text{disc}}(\mathcal{T}_h) = \{v,\ v|_{K_i} \in P_1(K_i),\ v(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin K_i\ \forall K_i \in \mathcal{T}_h\}$$

which is spanned by a set of $n = 3 \times N_e$ basis functions $\{\phi^1, \phi^2, \dots, \phi^n\} = \{\psi^1_{K_1}, \psi^2_{K_1}, \psi^3_{K_1}, \dots, \psi^1_{K_{N_e}}, \psi^2_{K_{N_e}}, \psi^3_{K_{N_e}}\}$, where there is a correspondence between the supraindex of ϕ^i and the supraindex and subindex of $\psi^r_{K_m}$, say $i = \text{iglob}(r, K_m)$. Each set $\{\psi^1_{K_m}, \psi^2_{K_m}, \psi^3_{K_m}\}$ is a a set of local basis functions on K_m, for which we have the set $\{\hat{\psi}^1_{\hat{K}}, \hat{\psi}^2_{\hat{K}}, \hat{\psi}^3_{\hat{K}}\}$ of functions defined on the master element $\hat{K}$ because both are affine equivalent, i.e.

$$\psi^r_{K_m}(\mathbf{x}) = \psi^r_{K_m}\left(F_{K_m}(\hat{\mathbf{x}})\right) = \hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}), \quad r = 1, 2, 3$$

Now, $\underline{\underline{A}}$ will be constructed by summing over all the elements, however, in this case the support of any function is a single element, so, if $\text{supp}\,(\phi^i) = \text{supp}\,(\phi^j) = K_m$, we have

$$A_{ij} = \int_{K_m} \phi^i|_{K_m}\, \phi^j|_{K_m}\, dK = \int_{K_m} \psi^{\,r}_{K_m}\, \psi^{\,s}_{K_m}\, dK = \int_{\hat{K}} \hat{\psi}^{\,r}_{\hat{K}}\, \hat{\psi}^{\,s}_{\hat{K}}\, |J_{K_m}|\, d\hat{K}$$

otherwise A_{ij} will be zero.

Exo. 2.16 *How the structure of matrix $\underline{\underline{A}}$ will be in the last case?*

- *Space of continous functions*

$$V_h(\mathcal{T}_h) = P_1(\mathcal{T}_h) = X(\mathcal{T}_h) \cap C^0(\Omega)$$

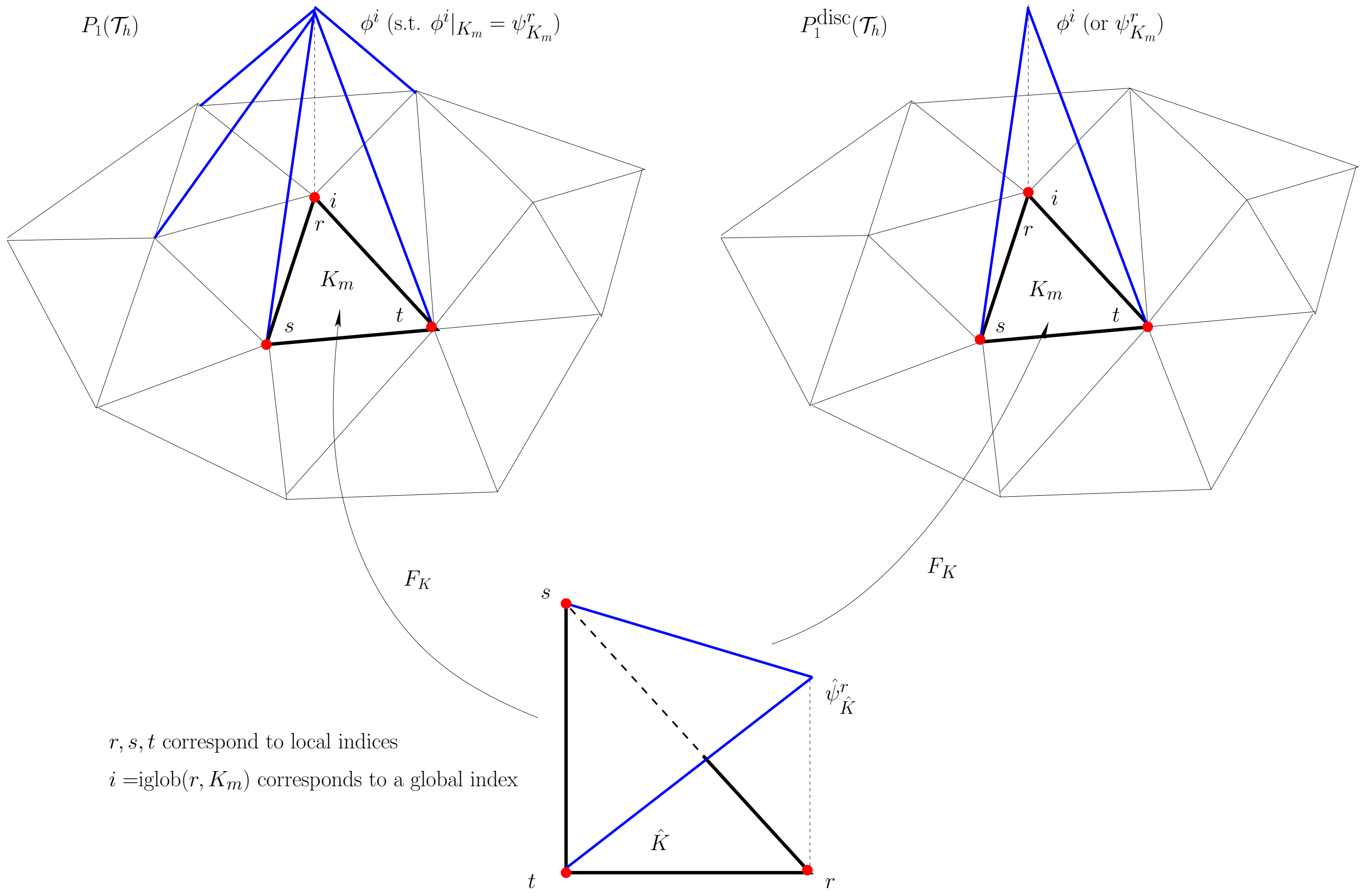
which is spanned by a set of $n = N_v$ basis functions $\{\phi^1, \phi^2, \dots, \phi^n\}$. Again, $\underline{\underline{A}}$ will be constructed by summing over all the elements. In this case the support of basis function ϕ^i are all the triangles that share vertex i, so, coefficient A_{ij} will be

$$A_{ij} = \sum_{\substack{K_m \in \\ (\text{supp}(\phi^i) \cap \\ \text{supp}(\phi^j))}} \int_{K_m} \phi^i|_{K_m} \, \phi^j|_{K_m} \, dK$$

but $\phi^i|_{K_m} = \psi^r_{K_m}$ and $\phi^j|_{K_m} = \psi^s_{K_m}$ for some r and s, then

$$\int_{K_m} \phi^i|_{K_m} \, \phi^j|_{K_m} \, dK = \int_{K_m} \psi^{\,r}_{K_m} \, \psi^{\,s}_{K_m} \, dK = \int_{\hat{K}} \hat{\psi}^{\,r}_{\hat{K}} \, \hat{\psi}^{\,s}_{\hat{K}} \, |J_{K_m}| \, d\hat{K}$$

As seen, in either case, what just need to compute elemental contributions to matrix $\underline{\underline{A}}$ *by integrating the basis functions defined on the master element* $\hat{K}$*.*


$P_1(\mathcal{T}_h)$
ϕ^i (s.t. $\phi^i|_{K_m} = \psi^r_{K_m}$)
$P_1^{\mathrm{disc}}(\mathcal{T}_h)$
ϕ^i (or $\psi^r_{K_m}$)
i
r
K_m
s
t
i
r
K_m
s
t
F_K
F_K
s
$\hat{\psi}^r_{\hat{K}}$
$\hat{K}$
t
r
r, s, t correspond to local indices
i =iglob(r, K_m) corresponds to a global index

Now consider the term involving the derivatives of the basis functions. We have

$$K_{ij} = \int_{\Omega} \nabla\phi^i(\mathbf{x}) \cdot \nabla\phi^j(\mathbf{x})\, d\Omega = \sum_{K_m \in \mathcal{T}_h} \int_{K_m} \nabla\phi^i(\mathbf{x})|_{K_m} \cdot \nabla\phi^j(\mathbf{x})|_{K_m}\, dK \tag{2.33}$$

Exo. 2.17 *Is the last operation legal for any* ϕ*?*

Once again, we transform the integral over K_m into an integral over $\hat{K}$, for which we need the previous result obtained in 2.29,

$$\int_{K_m} \nabla\phi^i(\mathbf{x})|_{K_m} \cdot \nabla\phi^j(\mathbf{x})|_{K_m}\, dK = \int_{K_m} \nabla\psi^r_{K_m}(\mathbf{x}) \cdot \nabla\psi^s_{K_m}(\mathbf{x})\, dK =$$

$$= \int_{\hat{K}} [B^{-T}_{K_m} \cdot \hat{\nabla}\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}})] \cdot [B^{-T}_{K_m} \cdot \hat{\nabla}\hat{\psi}^s_{\hat{K}}(\hat{\mathbf{x}})]\, |J_{K_m}|\, d\hat{K}$$

Again, we work with the local basis functions.

2.9.1 Numerical integration

Although, the master element $\hat{K}$ has a simpler shape, integrals are sometimes difficult to be perfomed exactly. Even when the coefficients of matrix $\underline{\underline{A}}$ involve the integration of polynomial functions, the right hand side may involve any function u:

$$F_i = \int_K u(\mathbf{x})\, \phi^i(\mathbf{x})|_K\, dK = \int_{\hat{K}} u(F_K(\hat{\mathbf{x}}))\, \psi^r_K(F_K(\hat{\mathbf{x}}))\, |J_K|\, d\hat{K} = \int_{\hat{K}} \hat{u}(\hat{\mathbf{x}})\, \hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}})\, |J_K|\, d\hat{K}$$

In theses cases we use numerical integration.

Def. 2.11 *Let* $\hat{K}$ *be non-empty compact connected subset. Let* n_g *be an integer. A quadrature on* $\hat{K}$ *with* n_g *points consists of:*

(i) A set of n_g *real numbers* $\{w_1, w_2, \ldots, w_{n_g}\}$ *called quadrature weights;*

(ii) A set of n_g *points* $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_{n_g}\}$ *called Gauss points or quadrature nodes.*

The largest integer such that

$$\forall \hat{p} \in \hat{P}_k \quad \int_{\hat{K}} \hat{p}(\hat{\mathbf{x}})\, d\hat{K} = \sum_{g=1}^{n_g} w_g\, \hat{p}(\hat{\mathbf{x}}_g)$$

is called the quadrature order and is denoted by r*. It can be shown that*

$$\frac{1}{meas(\hat{K})} \left| \int_{\hat{K}} f(\mathbf{x})\, d\hat{K} - \sum_{g=1}^{n_g} w_g\, \hat{f}(\hat{\mathbf{x}}_g) \right| \leq c\, h_{\hat{K}}^{r+1} \sup_{\hat{\mathbf{x}} \in \hat{K},\ |\alpha| = r+1} |D^\alpha f(\hat{\mathbf{x}})|$$

where $h_{\hat{K}}$ *is the* **diameter** *of* $\hat{K}$ *(the largest side) and* $c > 0$ *is a constant.*

Here we see again the practicity of working on the master element, since in this case we define the rules only once and for all.

These quadratures are tabulated. In 1D we have the so called Gauss-Legendre quadratures. Considering the master element being the interval $(-1, 1)$ quadratures of order $r = 3$, 5 and 7 are displayed in the table below. The quadrature points are zeros of Legendre polynomials.

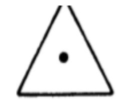
n	Points ξ_i	Weights w_i
2	0.5773502692	1.0000000000
	−0.5773502692	1.0000000000
3	0.7745966692	0.5555555556
	0.0000000000	0.8888888889
	−0.7745966692	0.5555555556
4	0.8611363116	0.3478548451
	0.3399810436	0.6521451549
	−0.3399810436	0.6521451549
	−0.8611363116	0.3478548451
5		
.		

- These rules can be adapted to other intervals rather than the reference one by simple change of variables

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}\,\xi\right)\,d\xi$$

- The cartesian product of 1D quadratures can be used in 2D and 3D so as to construct quadratures on quadrilateral and hexahedral elements.

$$\int_K f\,dx \sim f(a^{123})\,\text{area}(K)$$



r=1

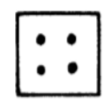
$$\int_K f\,dx \sim \sum_{j=1}^{n} f(b^j)\,\frac{\text{area}(K)}{3}$$

r=2

$$\int_K f\,dx \sim \sum_{j=1}^{3} \left[f(a^j)\,\frac{\text{area}(K)}{20} + f(b^j)\frac{2\,\text{area}(K)}{15}\right] + f(a^{123})\,\frac{9\,\text{area}(K)}{20}$$

r=3

$$\int_Q f\,dx \sim \left[f\left(\frac{h_1}{\sqrt{3}}, \frac{h_2}{\sqrt{3}}\right) + f\left(\frac{h_1}{\sqrt{3}}, -\frac{h_2}{\sqrt{3}}\right) + f\left(-\frac{h_1}{\sqrt{3}}, \frac{h_2}{\sqrt{3}}\right) + f\left(-\frac{h_1}{\sqrt{3}}, -\frac{h_2}{\sqrt{3}}\right)\right] \frac{\text{area}(K)}{4}.$$



r=3

2.10 1D Hermite elements

For the so called Lagrangian elements, proposition 2.10 shows how to construct a finite element (K, P_K, Σ_K) based on $(\hat{K}, \hat{P}, \hat{\Sigma})$. We used these elements to construct global spaces of functions associated to partitions of Ω made of intervals in 1D, triangles or quadrilaterals in 2D, etc.. When constructing such spaces we enforce some continuity restrictions at the interfaces between adjacent elements. In more general situations, when using as degrees of freedom the values of the derivates of the functions at a set of points, we need to add a modification which is called scaling. Since the master element is always the same but the real elements can have different sizes or shapes, if we want to preserve the derivatives of the functions at certain point when passing from $\hat{K}$ to different elements K's so as to create a space with C^1 continuity, we need to ensure that derivates are continuous at interlement boundaries.

We will restrict to the **1D case** for simplicity sake. Set $\hat{P} = P_3(\hat{K})$, being the master element $\hat{K} = (0, 1)$ and consider the degrees of freedom

$$\hat{\sigma}_1(\hat{v}) = \hat{v}(0), \quad \hat{\sigma}_2(\hat{v}) = \hat{v}'(0)$$

$$\hat{\sigma}_3(\hat{v}) = \hat{v}(1), \quad \hat{\sigma}_4(\hat{v}) = \hat{v}'(1)$$

The local basis functions are therefore

$$\hat{\psi}^1_{\hat{K}} = (2\,\hat{x} + 1)\,(\hat{x} - 1)^2, \quad \hat{\psi}^2_{\hat{K}} = \hat{x}\,(\hat{x} - 1)^2$$

$$\hat{\psi}^3_{\hat{K}} = (3 - 2\,\hat{x})\,\hat{x}^2, \quad \hat{\psi}^4_{\hat{K}} = (\hat{x} - 1)\,\hat{x}^2$$

This is called the Hermite element.

Exo. 2.18 *Check that* $\hat{\sigma}_i(\hat{\psi}^j_K) = \delta_{ij}$

Remember how we defined Σ_K: considering a basis of linear functionals $\hat{\sigma}_i \in \hat{\Sigma},\ i = 1, \ldots, n$, we define $\sigma_i \in \Sigma_K$ as:

$$\sigma_i(v) = \hat{\sigma}_i(\hat{v}) \ \ \forall\ \hat{v} \in \hat{P}$$

Instead of doing this, the idea is to perform a scaling, this is, define a set of coefficients $\alpha_i,\ i = 1, \ldots, n$ and the degrees of freedom $\sigma_i \in P_k$ such that

$$\sigma_i(v) = \alpha_i\, \hat{\sigma}_i(\hat{v}) \ \ \forall\ \hat{v} \in \hat{P}$$

In this way (K, P_K, Σ_K) will also be a finite element. Consider now a partition $\mathcal{T}_h$ of Ω, i.e., a 1D finite element mesh made of nonoverlapping intervals. In this case, we choose as coefficients α_i on each element K the followings:

$$\alpha_1 = \alpha_3 = 1, \quad \alpha_2 = \alpha_4 = \frac{1}{h_K}$$

where h_K is the size of element K. The local basis functions defined on K are therefore, for any $x \in K,\ x = F_K(\hat{x})$

$$\psi_K^1(x) = \hat{\psi}_{\hat{K}}^1(\hat{x}), \quad \psi_K^2(x) = h_K\, \hat{\psi}_{\hat{K}}^2(\hat{x})$$
$$\psi_K^3(x) = \hat{\psi}_{\hat{K}}^3(\hat{x}), \quad \psi_K^4(x) = h_K\, \hat{\psi}_{\hat{K}}^4(\hat{x})$$

We may now define a space of functions associated to this partition which will be C^1. The space is described by

$$V_h = \{v,\ v|_K \in P_3(K), \ \ \forall K \in \mathcal{T}_h\} \ \cap \ C^1(\Omega)$$

At node i, shared by the subintervals (or elements) K_l (left) and K_r (right), we define two basis functions

$$\phi^{i,0}(x) = \begin{cases} \psi_{K_l}^3 & \text{if } x \in K_l \\ \psi_{K_r}^1 & \text{if } x \in K_r \\ 0 & \text{otherwise} \end{cases}, \qquad \phi^{i,1}(x) = \begin{cases} \psi_{K_l}^4 & \text{if } x \in K_l \\ \psi_{K_r}^2 & \text{if } x \in K_r \\ 0 & \text{otherwise} \end{cases}$$

The set $\{\phi^{0,0}, \phi^{0,1}, \phi^{1,0}, \phi^{1,1}, \dots, \phi^{N_v,0}, \phi^{N_v,1}\}$, where N_v is the number of nodes, is a basis for V_h.

Exo. 2.19 *Show that a function* $v_h \in V_h$ *is defined by its value and that of its derivative at the nodes of* $\mathcal{T}_h$.

3 Projection of functions onto Finite element Spaces

We have to use the previous ingredients to solve our discrete variational problems. This is a good excuse to introduce a particular type of variational problem of interest to us, that is related to the problem of finding the best approximation of a function u from a finite element space V_h. In the next section we introduce these problems and show how they are computationally solved.

3.1 Definitions

We begin by recalling the definition of scalar or inner product.

Def. 3.1 Scalar product *Let V be a vector space. A mapping $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called a scalar product in V if for any f, g and $h \in V$ holds:*

(i) $(f, g) = (g, f)$ **(symmetry)**

(ii) $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$ **(linearity in the first argument)**

(iii) $(f, f) \geq 0$, $(f, f) = 0$ *if and only if* $f = 0$ **(positive definition)**

These inner products are actually bilinear forms.

Examples:

- $L^2(\Omega)$-inner product

$$(f, g)_{L^2(\Omega)} = \int_\Omega f\, g\, d\Omega$$

- $H^1(\Omega)$-inner product

$$(f,g)_{H^1(\Omega)} = \int_\Omega (f\,g + \nabla f \cdot \nabla g)\,d\Omega$$

Exo. 3.1 *Check that the $L^2(\Omega)$ and $H^1(\Omega)$ inner products satisfy the definition of inner product.*

Def. 3.2 Orthogonality*: Two vectors f and g are said to be orthogonal if $(f,g) = 0$*

We have already mentioned some norms that are induced by these inner products, (i.e. $\|x\| = [(x,x)]^{\frac{1}{2}}$):

- $L^2(\Omega)$-norm

$$\|f\|_{L^2(\Omega)} = \sqrt{\int_\Omega f^2\,d\Omega}$$

- $H^1(\Omega)$-norm

$$\|f\|_{H^1(\Omega)} = \sqrt{\int_\Omega (f^2 + \|\nabla f\|^2)\,d\Omega}$$

We can also define a **distance** $d(\cdot,\cdot)$ between elements of the space as:

$$d(f,g) = \|f-g\| = \sqrt{(f-g, f-g)}$$

3.1.1 The best approximation from a finite element space

Now, consider a space V and a finite dimensional subspace V_h like those finite element spaces introduced in the previous sections. Given a function $u \in V$, we want to find the best approximation to u from this subspace V_h. The ideia is to find $u_h \in V_h$ such that the distance to u is minimum.

Theorem 3.3 *There exists one and only one element $u_h \in V_h$ such that*

$$d(u, u_h) \leq d(u, v_h) \quad \forall v_h \in V_h \tag{3.1}$$

and u_h is the orthogonal projection of u over V_h.

We want to see how to construct u_h. So, let us suppose that u_h that satisfies (3.1) exists. In that case, the functional

$$j(s) = d(u, u_h + s\, v_h)^2 = \|u - (u_h + s\, v_h)\|^2 \tag{3.2}$$

will have a minimum at $s = 0$ $\forall v_h \in V_h$, i.e., if u_h minimizes $d(u, u_h)^2$, then $j(0) \leq j(s)$ $\forall s \in \mathbb{R}$. Using the linearity and symmetry of the inner product we show that

$$\begin{aligned} j(s) = d(u, u_h + s\, v_h)^2 &= (u - (u_h + s\, v_h), u - (u_h + s\, v_h)) = (u - u_h, u - u_h) - 2\, s\, (u - u_h, v_h) + s^2\, (v_h, v_h) \\ &= \|u - u_h\|^2 - 2\, s\, (u - u_h, v_h) + s^2\, \|v_h\|^2 \end{aligned}$$

Now, to have a minimum of this functional for all $v_h \in V_h$ when $s = 0$, its derivative with respect to s is necessarely equal to zero at that value of s, i.e.,

$$\frac{dj}{ds}(0) = -2\, (u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h$$

So u_h satisfies

$$(u_h, v_h) = (u, v_h) \quad \forall v_h \in V_h$$

This means that the difference between u and u_h is necessarely orthogonal to all $v_h \in V_h$.

Notice that this is a variational problem like those we have presented: "Determine $u_h \in V_h$, such that

$$a(u_h, v_h) = l(v_h)$$

holds for all $v_h \in V_h$". In this case, the bilinear form $a(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are based on the inner product:

$$\begin{cases} a(u_h, v_h) = (u_h, v_h) \\ \\ \ell(v_h) = (u, v_h) \end{cases}$$

Examples:

- *Best approximation in the $L^2(\Omega)$-norm:* "Determine $u_h \in V_h$, such that

  $$\int_\Omega u_h \, v_h \, d\Omega = \int_\Omega u \, v_h \, d\Omega$$

  holds for all $v_h \in V_h$". The solution u_h of this problem will minimize $\|u - v_h\|_{L^2(\Omega)}$ over V_h.

- *Best approximation in the $H^1(\Omega)$-norm:* "Determine $u_h \in V_h$, such that

  $$\int_\Omega (\nabla u_h \cdot \nabla v_h + u_h \, v_h) \, d\Omega = \int_\Omega (\nabla u \cdot \nabla v_h + u \, v_h) \, d\Omega$$

  holds for all $v_h \in V_h$". The solution u_h of this problem will minimize $\|u - v_h\|_{H^1(\Omega)}$ over V_h.

We are familiar with these problems, the only difference is that function u is given to us. We also know under which conditions the problems are well-posed so as their solution exists and is unique, we require:

(i) The linear functional ℓ to be continuous;

(ii) The bilinear form a to be continuous and strongly coercive;

For instance, for the $L^2(\Omega)$-inner product, it is easy to show that the bilinear form is strongly coercive, since

$$a(u_h, u_h) = \int_\Omega u_h^2 \, d\Omega = \|u_h\|^2_{L^2(\Omega)} \quad \Rightarrow \quad \frac{a(u_h, u_h)}{\|u_h\|^2_{L^2(\Omega)}} = 1 \; \forall u_h \in V_h$$

so, the coercivity constant is simply $\alpha = 1$.

Exo. 3.2 *What about the continuity of ℓ in the last case?*

Another way of seeing this is by looking at the associated linear system. Consider a basis $\{\phi^1, \phi^2, \dots, \phi^n\}$ of the finite dimensional space V_h and write u_h as linear combination of theses functions

$$u_h = \sum_{i=1}^{n} U_j \, \phi^j \tag{3.3}$$

we end up with a linear system of equations as already shown in previous lectures

$$\underline{\underline{A}}\,\underline{U} = \underline{F}$$

whose matrix $\underline{\underline{A}}$ ($A_{ij} = (\phi^i, \phi^j)$) is nonsingular. To prove that matrix $\underline{\underline{A}}$ is nonsingular for any scalar product on V_h we will show that $\underline{\underline{A}}$ is symmetric and positive definite. The symmetry is obvious because the inner product is symmetric by definition. Now, for any $w_h = \sum_i W_i \, \phi^i \in V_h, \;\; w_h \neq 0$ we have

$$\underline{W}^T \, \underline{\underline{A}}\,\underline{W} = \sum_{i,j} W_i \, A_{ij} W_j = \sum_{i,j} W_i \, (\phi^i, \phi^j) \, W_j = \left(\sum_i W_i \phi^i, \sum_j W_j \phi^j \right) = (w_h, w_h) > 0$$

then $\underline{\underline{A}}$ is positive definite.

For the $L^2(\Omega)$ best approximation we have

$$A_{ij} = \int_\Omega \phi^i(\mathbf{x})\,\phi^j(\mathbf{x})\,d\Omega, \qquad F_i = \int_\Omega u(\mathbf{x})\,\phi^i(\mathbf{x})\,d\Omega$$

For the $H^1(\Omega)$ best approximation we have

$$A_{ij} = \int_\Omega \left[\phi^i(\mathbf{x})\,\phi^j(\mathbf{x}) + \nabla\phi^i(\mathbf{x})\cdot\nabla\phi^j(\mathbf{x})\right]d\Omega, \qquad F_i = \int_\Omega \left[u(\mathbf{x})\,\phi^i(\mathbf{x}) + \nabla u(\mathbf{x})\cdot\nabla\phi^i(\mathbf{x})\right]d\Omega$$

Computation of these integrals we have already learned in previous sections by using the affine mapping. We will apply that to solve a few examples.

Exo. 3.3 *What about continuity and coercivity of a in the last case?*

3.2 Assembly of finite element matrices

In this section we explain the assembly process used to construct the linear system of equations associated to one of the variational problems described above.
The ingredients involved in this construction are:

1. A partition $\mathcal{T}_h$ of Ω made up of elements $K_m,\ m = 1, \ldots, N_{el}$ that are affine equivalent to a master element $\hat{K}$.

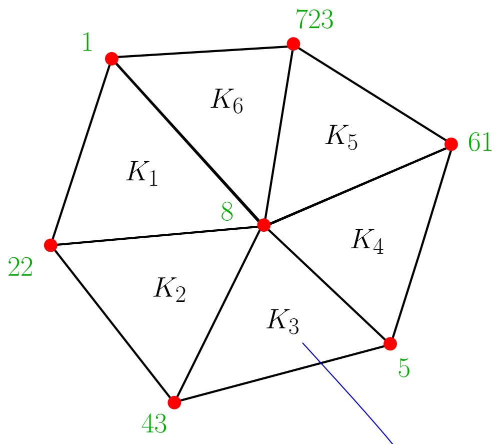
2. A space of functions associated to the partition: $V_h(\mathcal{T}_h), \mathrm{dim} V_h = N$ (it can be a space of totally discontinuous or continuous functions).

3. An incidence or **connectivity** matrix `conec` of dimension $N_{el} \times n_{loc}$ that describes the relation between the elements in $\mathcal{T}_h$ and the global unknowns (see figure below).

4. A quadrature rule on $\hat{K}$: $\{(w_g, \hat{\mathbf{x}}_g)\},\ g = 1, \ldots, n_g$.

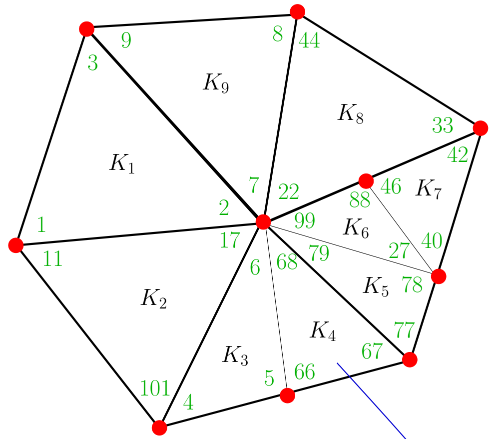
Given the function u to be projected over the space $V(\mathcal{T}_h)$, with these ingredients the global matrix $\underline{\underline{A}}$ denoted by `Aglo` below and the global right hand side vector $\underline{F}$ denoted by `RHS` can be assembled as shown in the pseudo-code below.

Conforming mesh



$$\texttt{conec} = \begin{bmatrix} 1 & 22 & 8 \\ 22 & 43 & 8 \\ 43 & 5 & 8 \\ 8 & 5 & 61 \\ 8 & 61 & 723 \end{bmatrix}$$

Nonconforming mesh



$$\texttt{conec} = \begin{bmatrix} 1 & 2 & 3 \\ 11 & 101 & 17 \\ 6 & 4 & 5 \\ 66 & 67 & 68 \\ 77 & 78 & 79 \\ 27 & 88 & 99 \\ 40 & 42 & 46 \\ 22 & 33 & 44 \\ 7 & 8 & 9 \end{bmatrix}$$

function [Aglo RHS] = Assembly(...)

  for $g = 1, \ldots, n_g$ **do** ▷ *Basis functions and derivatives at Gauss points on $\hat{K}$*

    Calculate $\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g)$ and $\hat{\nabla}\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g), \;\; r = 1, \ldots, n_{loc}$

  end for

  Initialize RHS and Aglo to zero

  for $m = 1, \ldots, N_{el}$ **do** ▷ *Loop over elements*

    Calculate $|J_{K_m}|$ and $B^{-T}_{K_m}$

    Initialize rhse and ae to zero

    for $g = 1, \ldots, n_g$ **do** ▷ *Loop over Gauss points*

      for $r = 1, \ldots, n_{loc}$ **do**

        $\texttt{rhse}(r) = \texttt{rhse}(r) + |J_{K_m}| * w_g * \left[u(F_K(\hat{\mathbf{x}}_g)) * \hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g) + \nabla u(F_K(\hat{\mathbf{x}}_g)) \cdot B^{-T}_{K_m} \cdot \hat{\nabla}\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g) \right]$

        for $s = 1, \ldots, n_{loc}$ **do**

          $\texttt{ae}(r, s) = \texttt{ae}(r, s) + |J_{K_m}| * w_g * \left[\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g) * \hat{\psi}^s_{\hat{K}}(\hat{\mathbf{x}}_g) + B^{-T}_{K_m} \cdot \hat{\nabla}\hat{\psi}^r_{\hat{K}}(\hat{\mathbf{x}}_g) \cdot B^{-T}_{K_m} \cdot \hat{\nabla}\hat{\psi}^s_{\hat{K}}(\hat{\mathbf{x}}_g) \right]$

        end for

      end for

    end for ▷ *End loop over Gauss points*

    for $r = 1, \ldots, n_{loc}$ **do** ▷ *Assembly elementary matrix into global matrix*

      $I = \texttt{conec}(K_m, r)$

      $\texttt{RHS}(I) = \texttt{RHS}(I) + \texttt{rhse}(r)$

      for $s = 1, \ldots, n_{loc}$ **do**

        $J = \texttt{conec}(K_m, s)$

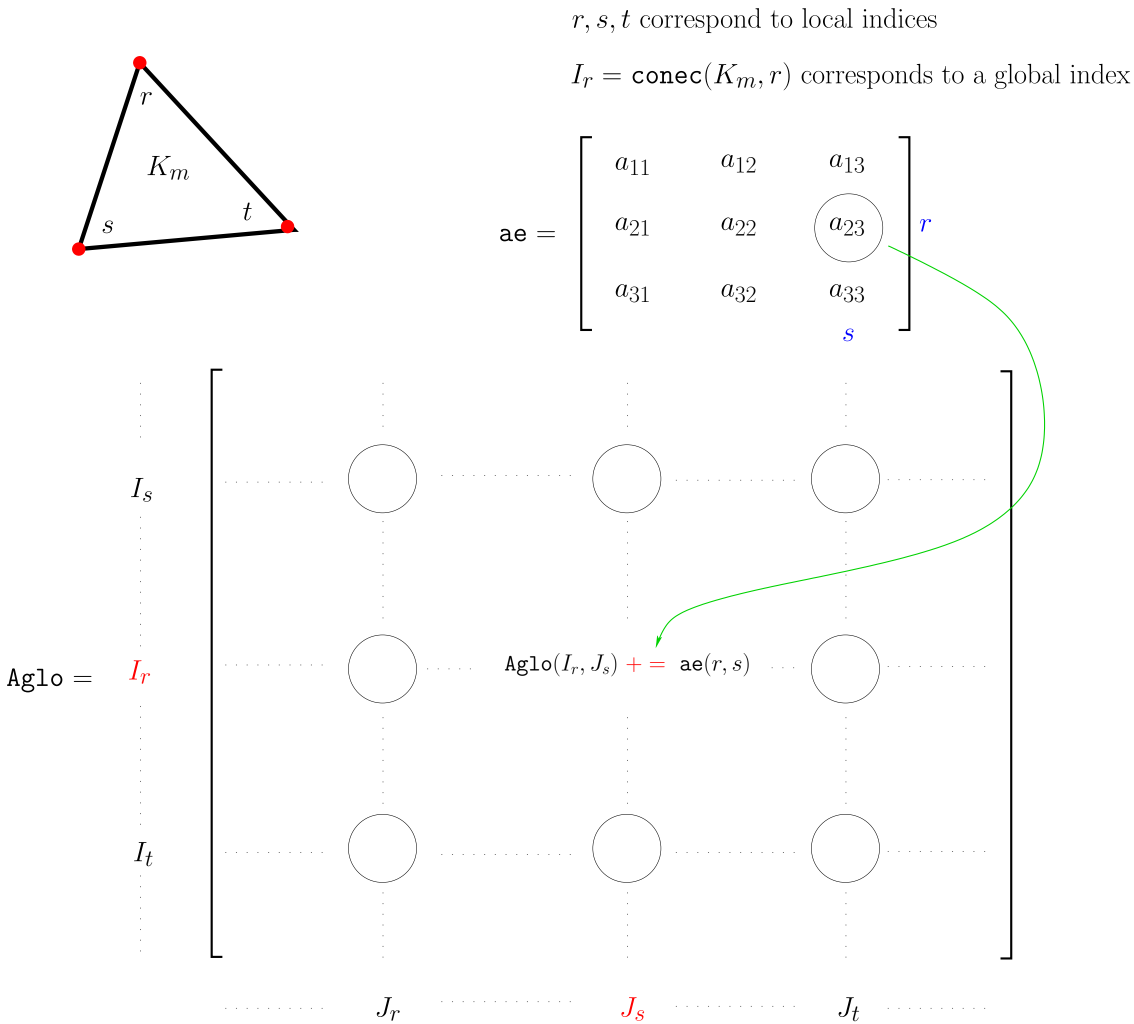
        $\texttt{Aglo}(I, J) = \texttt{Aglo}(I, J) + \texttt{ae}(r, s)$

      end for

    end for

  end for ▷ *End loop over elements*

end function


r, s, t correspond to local indices
I_r = conec(K_m, r) corresponds to a global index
r
K_m
s
t
ae =
a11 a12 a13
a21 a22 a23
a31 a32 a33
r
s
Aglo =
I_s
I_r
I_t
Aglo(I_r, J_s) += ae(r, s)
J_r
J_s
J_t

3.2.1 Examples

In the following examples we find the best approximation from different spaces to function $u(x) = e^{4\,(x-0.5)^2}$.

(i) u **continuous** and V_h made up of **piecewise continuous** functions;

(ii) u **continuous** and V_h made up of **continuous** functions;

(iii) u **piecewise continuous** and V_h made up of **continuous** functions;

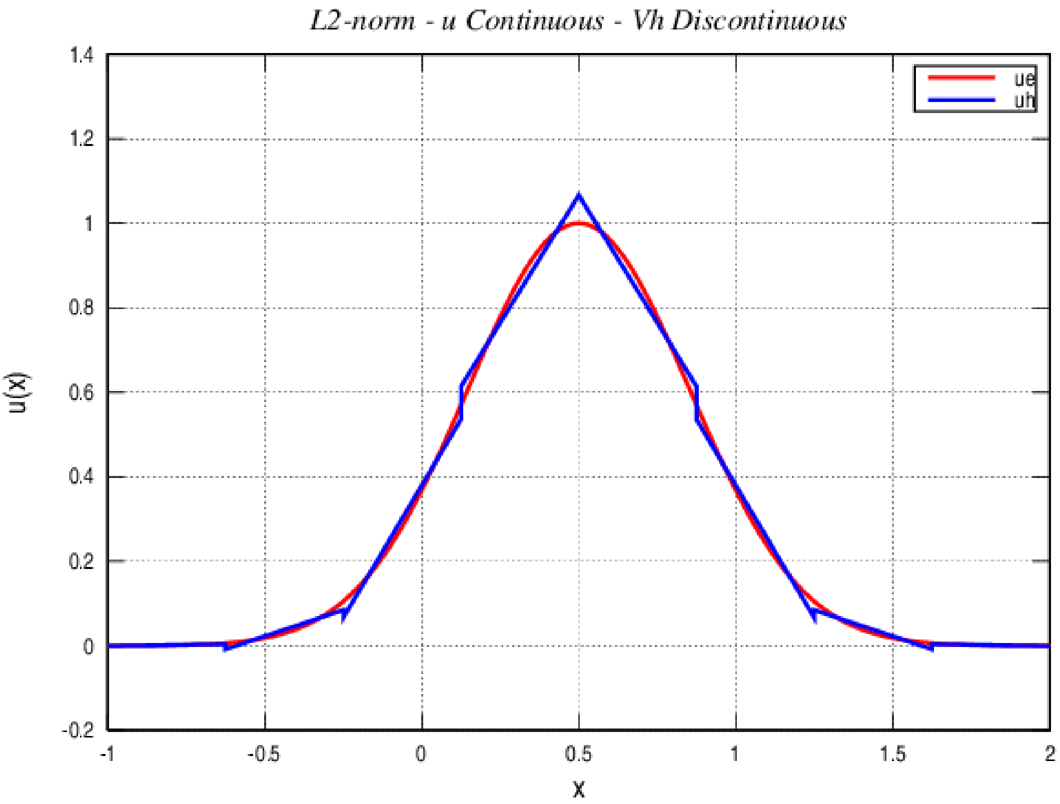
The last one is a common procedure to post-process a function $w_h \in V_h(\mathcal{T}_h) \in H^1(\Omega)$. Let us suppose we have

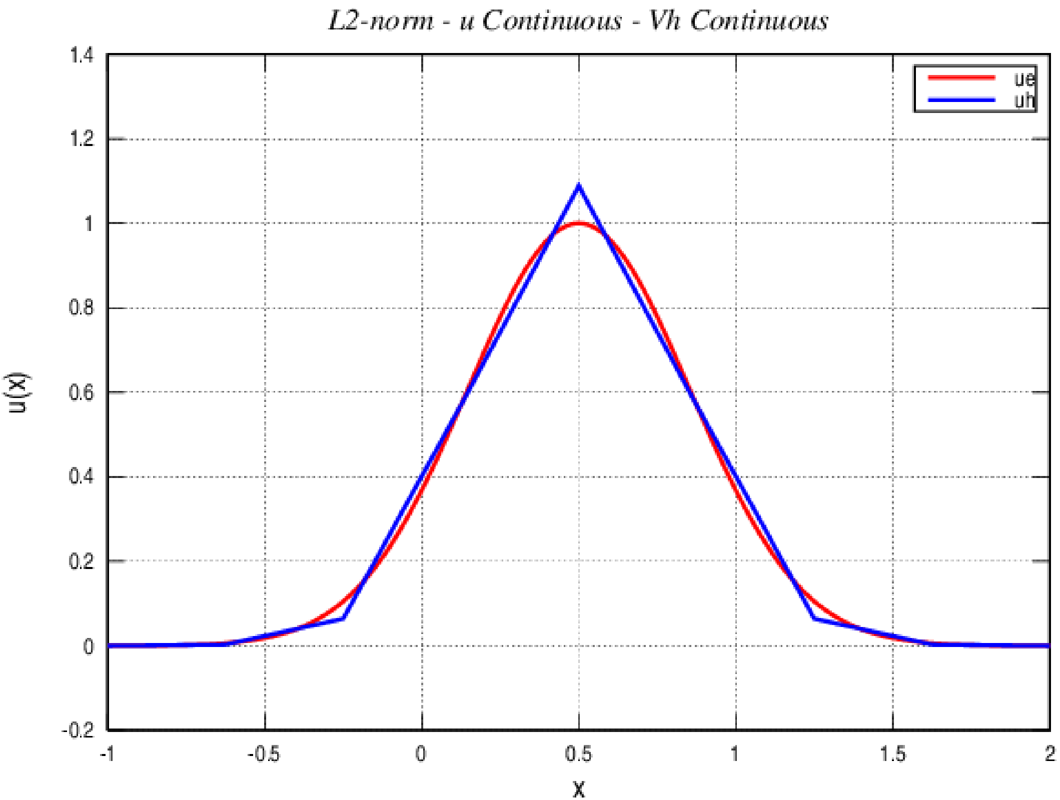
$$w_h(\mathbf{x}) = \sum_{i=1}^{n} W_j\, \phi^j(\mathbf{x})$$

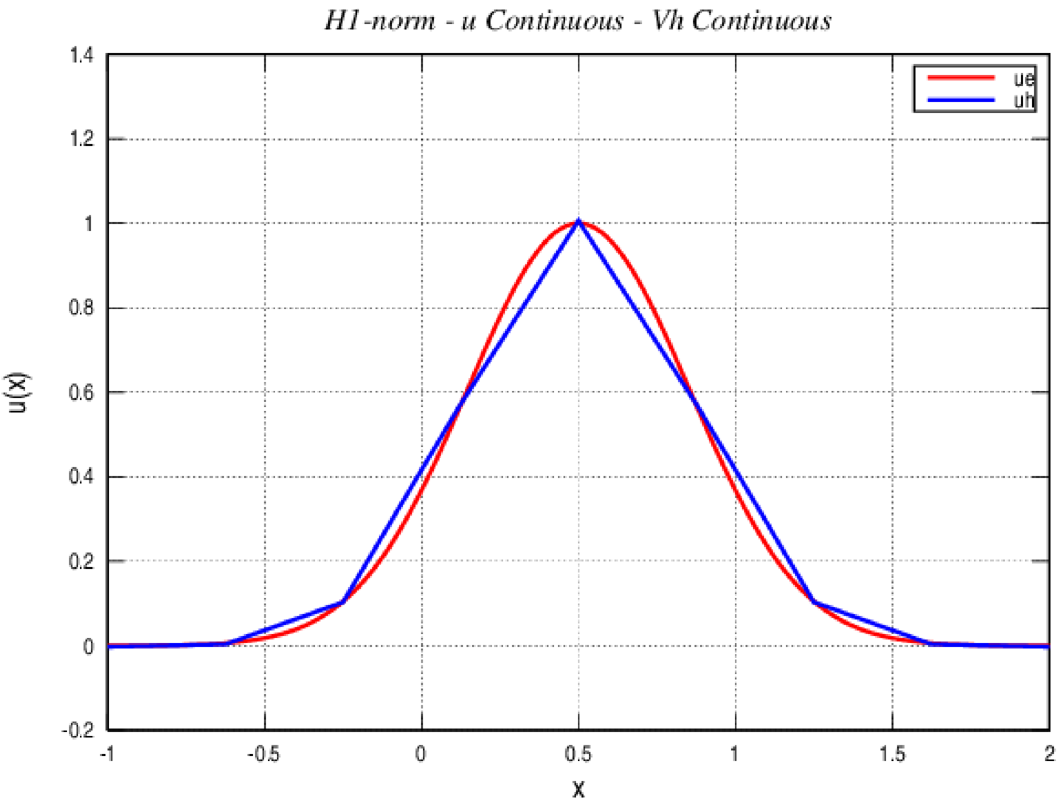
its gradient

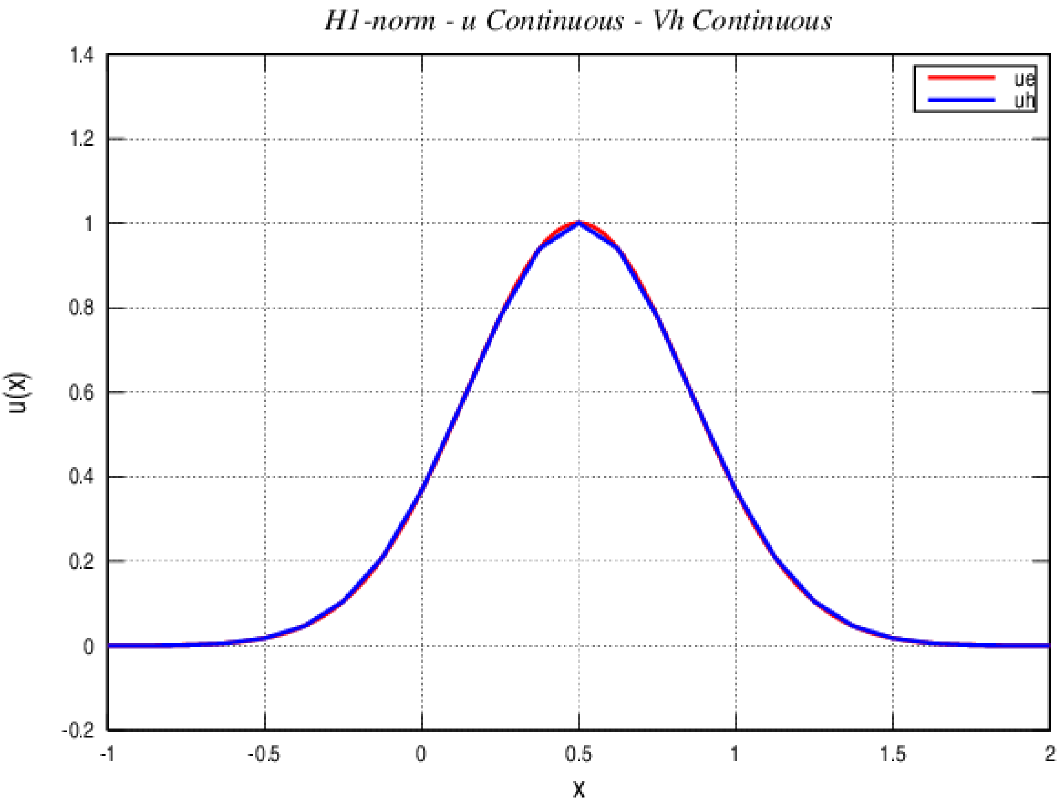
$$\nabla w_h(\mathbf{x}) = \sum_{i=1}^{n} W_j\, \nabla\phi^j(\mathbf{x})$$

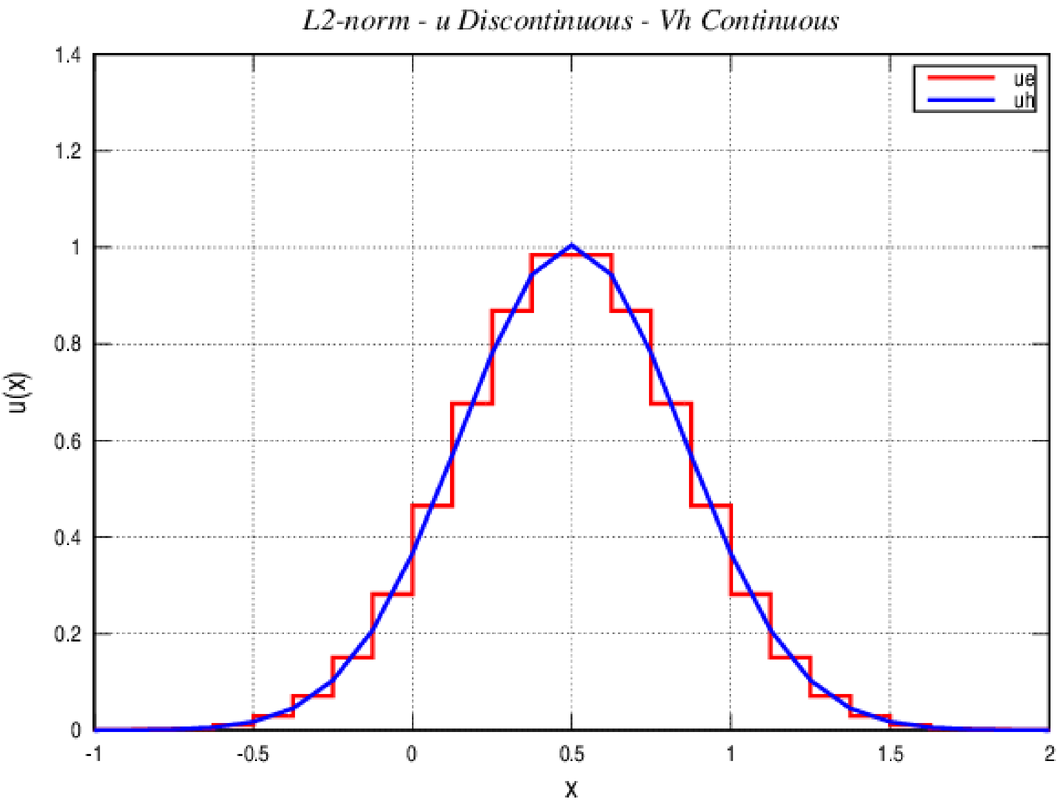
will be only elementwise continuous, so the derivatives are not necessarely continuous at the inter-element nodes/edges. In this case, we perform the so called **gradient recovery** in order to find an approximation of this gradient from a space of continuous functions. In this last example we construct an elementwise constant function whose value on each element is made equal to the function u used above evaluated at the midpoint of the corresponding element. The result of projecting such function is illustrated below.


L2-norm - u Continuous - Vh Discontinuous
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
u(x)
-1
-0.5
0
0.5
1
1.5
2
x
ue
uh


L2-norm - u Continuous - Vh Continuous
ue
uh
u(x)
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.5
0
0.5
1
1.5
2


H1-norm - u Continuous - Vh Continuous
ue
uh
u(x)
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.5
0
0.5
1
1.5
2


H1-norm - u Continuous - Vh Continuous
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
u(x)
-1
-0.5
0
0.5
1
1.5
2
x
ue
uh



L2-norm - u Discontinuous - Vh Continuous
ue
uh
u(x)
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
1.5
2

Exo. 3.4 *Using the code provided in the classroom:*

(i) *Play with the different options available in the code to project a function.*

(ii) *Add the necessary lines to compute the error in the* L^2 *and* H^1 *norms.*

(iii) *Using the previous item perform a mesh refinement study and plot the error as function of the mesh size.*

(iv) *(Optional) Program the P2 element.*

4 Local estimates of interpolation error

Remember the best approximation property (Céa's lemma)

Lemma 4.1 *If $a(\cdot,\cdot)$ and $\ell(\cdot)$ are continuous in V and $a(\cdot,\cdot)$ is strongly coercive, then*

$$\|u - u_h\|_V \le \frac{N_a}{\alpha}\,\|u - v_h\|_V \qquad \forall\, v_h \in V_h \tag{4.1}$$

in other words $\|u - u_h\|_V \le C \inf_{v_h \in V_h} \|u - v_h\|_V$.

In order to prove convergence of u_h to u we have to choose appropriate finite element spaces V_h and show that there are good approximations to u from V_h. The usual way of doing this is by using an interpolant $\mathcal{I}_h u \in V_h$ (e.g. the Lagrange interpolant), since

$$\|u - u_h\|_V \le C\,\|u - \mathcal{I}_h u\|_V \tag{4.2}$$

and the idea is to study how $\|u - \mathcal{I}_h u\|_V$ behaves.

So, the plan for the following sections is to introduce:

- The interpolation operators;
- The local estimates of the interpolation error;
- The global error estimates;
- Some issues about regularity of meshes;

4.1 The local interpolation operator

The local interpolation operator $\mathcal{I}_K : V(K) \to P_K$ is defined as

$$\mathcal{I}_K v = \sum_{i=1}^{n} \sigma_i(v)\, \psi^i \quad \forall v \in V(K) \tag{4.3}$$

$\mathcal{I}_K$ has the following properties:

(i) It is linear;

(ii) It is a projection: $\mathcal{I}_K(p) = p \;\; \forall p \in P_K$ (P_K is invariant under $\mathcal{I}_K$).

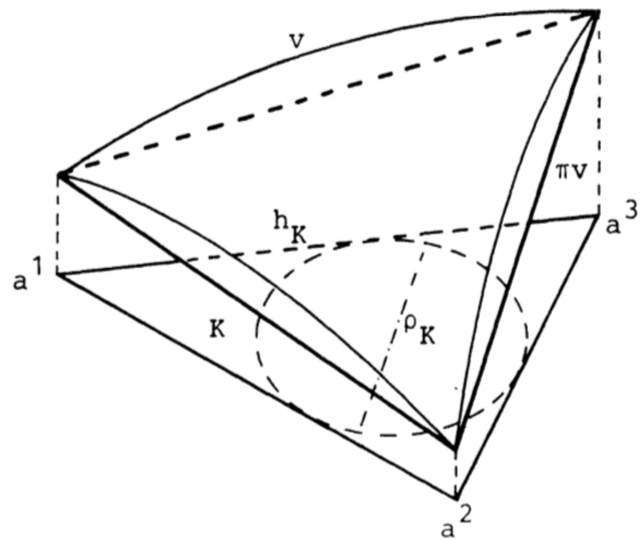
Exo. 4.1 *Prove the previous properties.*

Exo. 4.2 *For the P_1-Lagrange element, show that* $\sum_{j=1}^{3} \psi^j(\mathbf{x}) = 1$ *and* $x_k = \sum_{j=1}^{3} X_k^j \psi^j(\mathbf{x})$.

4.2 Local error estimates in L^∞

For simplicity sake **we consider from now on P_1 triangular elements**. This is enough to introduce the main ingredients involved and several important results. The idea is to study at the element level the difference between u and its interpolant $\mathcal{I}_K u$ (see figure below).

In the figure we have h_K, which is the **diameter** of K (the largest side) and ρ_K which is the inner diameter of K (the largest ball inscribed in K).



Now, we can present an importan theorem:

Theorem 4.2 *Let K be a P_1-triangle, h_K its diameter and ρ_K its inner diameter. Then, for all $v \in C^\infty$,*

(a) $\|v - \mathcal{I}_K v\|_{L^\infty(K)} \leq 2\, h_K^2 \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)}$

(b) $\max_{|\alpha|=1} \|D^\alpha (v - \mathcal{I}_K v)\|_{L^\infty(K)} \leq 6\, \dfrac{h_K^2}{\rho_K} \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)}$

Proof. Let $\{\psi^1, \psi^2, \psi^3\}$ be the basis for $P_1(K)$ and let $\mathbf{x}^j = (X_1^j, X_2^j)$ be the position of the j-th node of the element, then

$$\mathcal{I}_K v(\mathbf{x}) = \sum_{i=1}^{3} v(\mathbf{x}^j)\, \psi^j(\mathbf{x}), \quad x \in K \tag{4.4}$$

Now, we perform a Taylor expansion around $\mathbf{x} \in K$

$$v(\mathbf{y}) = v(\mathbf{x}) + \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x})\,(y_k - x_k) + R(\mathbf{x}, \mathbf{y}), \tag{4.5}$$

where the rest R is:

$$R(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \sum_{k,\ell}^{2} \frac{\partial^2 v}{\partial x_k \partial x_\ell}(\boldsymbol{\xi})\,(y_k - x_k)\,(y_l - x_l) \tag{4.6}$$

and $\boldsymbol{\xi}$ is a point on the line segment between $\mathbf{x}$ and $\mathbf{y}$. Now, evaluate the expansion at $\mathbf{y} = \mathbf{x}^j$

$$v(\mathbf{x}^j) = v(\mathbf{x}) + p^j(\mathbf{x}) + R^j(\mathbf{x}), \tag{4.7}$$

where $p^j(\mathbf{x}) = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x})\,(X_k^j - x_k)$ and $R^j(\mathbf{x}) = R(\mathbf{x}, \mathbf{x}^j)$.
Now, since $\|X_i^j - x_i\| \leq h_K, \quad j = 1, 2, 3, \quad i = 1, 2$, we have

$$|R^j(\mathbf{x})| \leq 2\, h_K^2 \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)} \tag{4.8}$$

Inserting $v(\mathbf{x}^j)$ into the definition of the interpolant:

$$\mathcal{I}_K v(\mathbf{x}) = \sum_{j=1}^{3} v(\mathbf{x})\,\psi^j(\mathbf{x}) + \sum_{j=1}^{3} p^j(\mathbf{x})\,\psi^j(\mathbf{x}) + \sum_{j=1}^{3} R^j(\mathbf{x})\,\psi^j(\mathbf{x}) \tag{4.9}$$

Let us consider each term separately:

$$\sum_{j=1}^{3} v(\mathbf{x})\,\psi^j(\mathbf{x}) = v(\mathbf{x})\,\sum_{j=1}^{3} \psi^j(\mathbf{x}) = v(\mathbf{x}) \tag{4.10}$$

since $\sum_{j=1}^{3} \psi^j(\mathbf{x}) = 1$ (see **Exo. 4.2**). For the second term

$$\sum_{j=1}^{3} p^j(\mathbf{x})\,\psi^j(\mathbf{x}) = \sum_{j=1}^{3}\sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x})(X_k^j - x_k)\,\psi^j(\mathbf{x}) = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x})\left\{\sum_{j=1}^{3} X_k^j \psi^j(\mathbf{x}) - x_k \sum_{j=1}^{3} \psi^j(\mathbf{x})\right\} \tag{4.11}$$

But, we can show that $\sum_{j=1}^{3} X_k^j \psi^j(\mathbf{x}) = x_k$ (see **Exo. 4.2**), therefore the second term vanishes. We have then

$$\mathcal{I}_K v(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{3} R^j(\mathbf{x})\,\psi^j(\mathbf{x}) \tag{4.12}$$

and thus

$$|v(\mathbf{x}) - \mathcal{I} v_K(\mathbf{x})| \leq \max_j |R^j(\mathbf{x})| \sum_{j=1}^{3} \psi^j(\mathbf{x}) = \max_j |R^j(\mathbf{x})| \leq 2\,h_K^2 \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)} \tag{4.13}$$

implying assertion **(a)**. Now, by differentiating $\mathcal{I}_K v(\mathbf{x}) = \sum_{i=1}^{3} v(\mathbf{x}^j)\,\psi^j(\mathbf{x})$ with respect to x_m and using the Taylor expansion again evaluated at $\mathbf{x}^j$ we obtain

$$\frac{\partial \mathcal{I}_K v}{\partial x_m}(x) = \sum_{j=1}^{3} v(x)\, \frac{\partial \psi^j}{\partial x_m}(x) + \sum_{j=1}^{3} p^j(x)\, \frac{\partial \psi^j}{\partial x_m}(x) + \sum_{j=1}^{3} R^j(x)\, \frac{\partial \psi^j}{\partial x_m}(x) \tag{4.14}$$

On the right-hand side, the first term vanishes. The second term is

$$\begin{aligned}\sum_{j=1}^{3} p^j(\mathbf{x})\, \frac{\partial \psi^j}{\partial x_m}(\mathbf{x}) = \sum_{j,k} \frac{\partial v}{\partial x_k}(\mathbf{x})\, (X_k^j - x_k)\, \frac{\partial \psi^j}{\partial x_m}(\mathbf{x}) = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x}) \left[\sum_{j=1}^{3} X_k^j \frac{\partial \psi^j}{\partial x_m}(\mathbf{x}) - x_k \sum_{j=1}^{3} \frac{\partial \psi^j}{\partial x_m}(\mathbf{x}) \right] = \\ = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x}) \frac{\partial}{\partial x_m} \sum_{j=1}^{3} X_k^j \psi^j(\mathbf{x}) = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x}) \frac{\partial}{\partial x_m} x_k = \sum_{k=1}^{2} \frac{\partial v}{\partial x_k}(\mathbf{x}) \delta_{km} = \frac{\partial v}{\partial x_m}(\mathbf{x})\end{aligned} \tag{4.15}$$

Finally, taking the absolute value and using previous results for $|R^j(\mathbf{x})|$

$$\left| \frac{\partial v}{\partial x_m}(\mathbf{x}) - \frac{\partial \mathcal{I}_K v}{\partial x_m}(\mathbf{x}) \right| = \left| \sum_{j=1}^{3} R^j(\mathbf{x})\, \frac{\partial \psi^j}{\partial x_m}(\mathbf{x}) \right| \leq \max_j |R^j(\mathbf{x})| \sum_{j=1}^{3} \left| \frac{\partial \psi^j}{\partial x_m} \right| \leq 6\, \frac{h_K^2}{\rho_K} \max_{|\alpha|=2} \|D^\alpha v\|_{L^\infty(K)} \tag{4.16}$$

since $\left| \frac{\partial \psi^j}{\partial x_m} \right| \leq \frac{1}{\rho_K}$ (by looking at the figure in the beginning of this section, the reader can convince himself that the derivative of a P_1 function which equals 1 at a given node and is zero on the oposite side can never be grater than $1/\rho_K$).

□

4.3 Local error estimates in Sobolev spaces $W^{s,p}$

The previous theorem gives estimates of the interpolation error in the $L^\infty(K)$-norm, but what we need are estimates in $L^2(K)$ and $H^1(K)$. The aim is to be able to show the following important theorem:

Theorem 4.3 *Let (K, P_K, Σ_K) be a finite element, affine-equivalent to a master element $(\hat{K}, \hat{P}, \hat{\Sigma})$. Assume, $\hat{P} = P_1(\hat{K})$. Then, there exists a constant $C > 0$, independent of K, such that*

(a) $\|v - \mathcal{I}_K v\|_{L^2(K)} \leq C\, h_K^2\, |v|_{H^2(K)} \quad \forall v \in H^2(K)$

(b) $\|v - \mathcal{I}_K v\|_{H^1(K)} \leq C\, \dfrac{h_K^2}{\rho_K} |v|_{H^2(K)} \quad \forall v \in H^2(K)$

Using this results and summing over the elements we can show that the global interpolation error is:

$$\|v - \mathcal{I}_h v\|_{L^2(\Omega)} \leq C\, h^2 \sum_K |v|_{H^2(K)}$$

This will be the topic of the following lecture.