Introduction to the Finite Element method

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2 Finite Element Spaces

2.1 Introduction

A numerical method is a Galerkin finite element method if:

1. It is based on a variational formulation, i.e., Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\, v_h \in V_h \tag{2.1}$$

   where $a : V_h \times V_h \to \mathbb{R}$ is linear in its second argument and $\ell : V_h \to \mathbb{R}$ is a linear functional. Both, a and ℓ corresponds to the exact problem.

2. The discrete space V_h is a **finite element space.**

As an example, consider the 1D model problem previously introduced: "Determine $u_h \in V_h \subset H^1(0,1)$, such that $u_h(0) = 0$ and that

$$\int_0^1 (u_h'\, v_h' + \theta\, u_h\, v_h)\; dx = \int_0^1 f\, v_h\; dx \tag{2.2}$$

holds for all $v_h \in V_h$ satisfying $v_h(0) = 0$."

In the following sections the aim is to construct finite element spaces V_h to solve this problem. We begin with a few simple examples, introduce the concept of **degrees of freedom** and also some classical **finite element** basis.

2.2 1D examples

2.2.1 A space of polynomial functions in (a, b)

Consider the interval (a, b). We define the space as

$$V_h = P_k(a, b) = \{v, v = \sum_{i=0}^{k} \alpha_k\, x^k\} \tag{2.3}$$

e.g. $k = 1$

$$P_1(a, b) = \{v, v = \alpha + \beta\, x\} \tag{2.4}$$

This space has dimension 2. Once we have defined the space, we proceed like this:

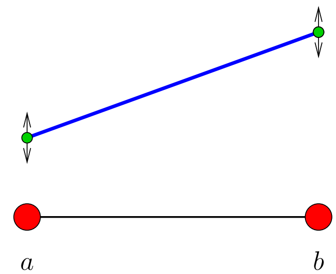
1. Define a set of degrees of freedom $\{\sigma_1, \sigma_2\}$ (i.e., a set of linear functionals of P_k in $\mathbb{R}$).
2. Define $\{\phi^1(x), \phi^2(x)\}$ by the relation: $\sigma_i(\phi^j) = \delta_{ij}$
   (*this is the Kronecker delta property, i.e.,* $\delta_{ij} = 1$ *if* $i = j$ *and* 0 *otherwise).*

For instance, we can consider as degrees of freedom the value of the function at the end points of the interval:

$$\sigma_1(v) = v(a) \tag{2.5}$$
$$\sigma_2(v) = v(b) \tag{2.6}$$

To compute the basis, consider functions $\phi^j(x) = \alpha_j + \beta_j\, x$. In order to find α_j and β_j, $\;j = 1, 2$ we have two 2×2 systems to solve:

$$P_1(a,b)$$



$$\sigma_1(\phi^1) = \alpha_1 + \beta_1\, a = 1, \quad \sigma_1(\phi^2) = \alpha_2 + \beta_2\, a = 0$$
$$\sigma_2(\phi^1) = \alpha_1 + \beta_1\, b = 0, \quad \sigma_2(\phi^2) = \alpha_2 + \beta_2\, b = 1$$

Therefore, the basis is:

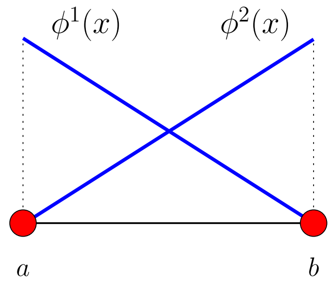
$$\phi^1(x) = \frac{b-x}{b-a}, \quad \phi^2(x) = \frac{x-a}{b-a} \tag{2.7}$$

The last choice seemed arbitrary, but it is a very **practical one**. If we want to describe a function in that space, the only thing I need is the value of the function at $x^1 = a$ and $x^2 = b$, since $\phi^i(x^j) = \delta_{ij}$, i.e.

$$v(x) = \sum_{i=1}^{2} U^j\, \phi^j(x) = \underbrace{U^1}_{v(a)}\, \phi^1(x) + \underbrace{U^2}_{v(b)}\, \phi^2(x) = v(a)\, \phi^1(x) + v(b)\, \phi^2(x)$$

As an exercise, compute the matrix $\underline{\underline{A}}$ for our model problem, i.e.

$$A_{ij} = a(\phi^i, \phi^j), \quad i, j = 1, 2$$



We will write $\underline{\underline{A}}$ as sum of two matrices

$$\underline{\underline{A}} = \underline{\underline{K}} + \theta\, \underline{\underline{M}} \tag{2.8}$$

where

$$K_{ij} = a_d(\phi_i, \phi_j) = \int_a^b (\phi^i)'(\phi^j)'\, dx, \qquad M_{ij} = a_r(\phi_i, \phi_j) = \int_a^b \phi^i\, \phi^j\, dx$$

calculating the integrals

$$K_{11} = a_d(\phi^1, \phi^1) = \int_a^b (\phi^1)'(\phi^1)'\, dx = \frac{1}{b-a},$$

$$K_{12} = K_{21} = a_d(\phi^2, \phi^1) = \int_a^b (\phi^2)'(\phi^1)'\, dx = -\frac{1}{b-a}$$

$$K_{22} = a_d(\phi^2, \phi^2) = \int_a^b (\phi^2)'(\phi^2)'\, dx = \frac{1}{b-a}$$

and similarly we compute

$$M_{11} = a_r(\phi^1, \phi^1) = \int_a^b \phi^1\phi^1\, dx = \frac{b-a}{3}$$

$$M_{12} = M_{21} = a_r(\phi^1, \phi^2) = \int_0^1 \phi^1 \phi^2 \, dx = \frac{b-a}{6}$$

and so on ..., finally giving

$$\underline{\underline{A}} = \begin{bmatrix} \frac{1}{b-a} & -\frac{1}{b-a} \\ \\ -\frac{1}{b-a} & \frac{1}{b-a} \end{bmatrix} + \theta \begin{bmatrix} \frac{b-a}{3} & \frac{b-a}{6} \\ \\ \frac{b-a}{6} & \frac{b-a}{3} \end{bmatrix}$$

All these computations at individual intervals or elements will be useful later on when we construct aproximations on spaces defined on a collection of such elements.

Exo. 2.1 *Compute the basis when we choose as degrees of freedom:*

$$\sigma_1(v) = v\left(\frac{a+b}{2}\right) \tag{2.9}$$

$$\sigma_2(v) = v'\left(\frac{a+b}{2}\right) \tag{2.10}$$

Exo. 2.2 *Consider the space* $P_2(a,b) = \{v, v = \alpha_0 + \alpha_1 x + \alpha_2 x^2\}$ *. Compute the basis* $\{\phi^1, \phi^2, \phi^3\}$ *when we choose as degrees of freedom*

$$\sigma_1(v) = v(a) \tag{2.11}$$

$$\sigma_2(v) = v\left(\frac{a+b}{2}\right) \tag{2.12}$$

$$\sigma_3(v) = v(b) \tag{2.13}$$

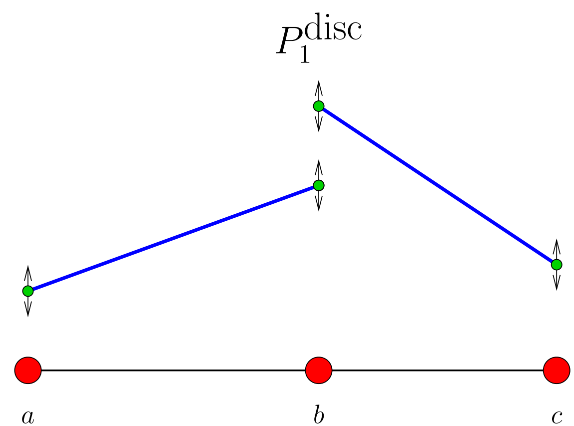
2.2.2 A polynomial space by parts

Consider the intervals (a, b) and (b, c) and define the space

$$V_h = P_k^{\text{disc}} = \{v,\ v|_{(a,b)} \in P_k(a,b),\quad v|_{(b,c)} \in P_k(b,c)\}$$

Consider again the case $k = 1$ for simplicity. The space has dimension 4. This is more or less evident if we notice that functions in this space are:

$$v = \begin{cases} \alpha + \beta\, x & \text{if } x \in (a,b) \\ \gamma + \epsilon\, x & \text{if } x \in (b,c) \end{cases}$$



Notice that such functions are not necessarily continuous at $x = b$ and therefore we have 4 degrees of freedom. For instance, choose for them:

$$
\begin{aligned}
\sigma_1(v) &= v(a) \\
\sigma_2(v) &= v(b^-) \\
\sigma_3(v) &= v(b^+) \\
\sigma_4(v) &= v(c)
\end{aligned}
$$

Now, we can compute the basis by using the relation $\sigma_i(\phi^j) = \delta_{ij}$. We write the basis function as

$$
\phi^j(x) = \begin{cases} \alpha_j + \beta_j\, x & \text{if } x \in (a,b) \\ \gamma_j + \epsilon_j\, x & \text{if } x \in (b,c) \end{cases}
$$

For $j = 1, \ldots, 4$ we have

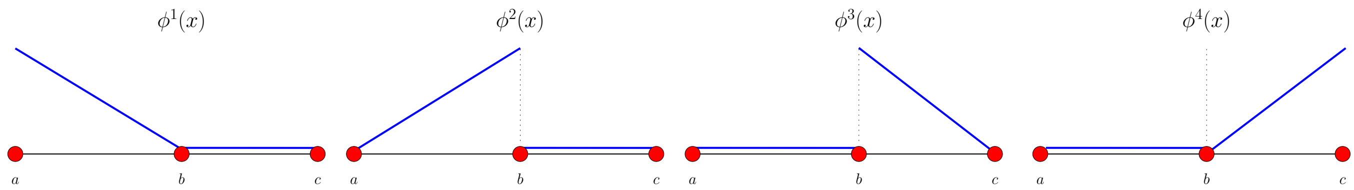
$$
\begin{aligned}
\sigma_1(\phi^j) &= \alpha_j + \beta_j\, a &= \delta_{1j} \\
\sigma_2(\phi^j) &= \alpha_j + \beta_j\, b^- &= \delta_{2j} \\
\sigma_3(\phi^j) &= \gamma_j + \epsilon_j\, b^+ &= \delta_{3j} \\
\sigma_4(\phi^j) &= \gamma_j + \epsilon_j\, c &= \delta_{4j}
\end{aligned}
$$

$$
\begin{cases} \alpha_1 + \beta_1\, a & = 1 \\ \alpha_1 + \beta_1\, b & = 0 \\ \gamma_1 + \epsilon_1\, b & = 0 \\ \gamma_1 + \epsilon_1\, c & = 0 \end{cases}, \quad
\begin{cases} \alpha_2 + \beta_2\, a & = 0 \\ \alpha_2 + \beta_2\, b & = 1 \\ \gamma_2 + \epsilon_2\, b & = 0 \\ \gamma_2 + \epsilon_2\, c & = 0 \end{cases}, \quad
\begin{cases} \alpha_3 + \beta_3\, a & = 0 \\ \alpha_3 + \beta_3\, b & = 0 \\ \gamma_3 + \epsilon_3\, b & = 1 \\ \gamma_3 + \epsilon_3\, c & = 0 \end{cases}, \quad
\begin{cases} \alpha_4 + \beta_4\, a & = 0 \\ \alpha_4 + \beta_4\, b & = 0 \\ \gamma_4 + \epsilon_4\, b & = 0 \\ \gamma_4 + \epsilon_4\, c & = 1 \end{cases},
$$

By inspection we find that the basis is:

$$\phi^1(x) = \begin{cases} \dfrac{b-x}{b-a} & \text{if } x \in (a,b) \\ \\ 0 & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^2(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in (a,b) \\ \\ 0 & \text{if } x \in (b,c) \end{cases}$$

$$\phi^3(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \\ \dfrac{c-x}{c-b} & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^4(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \\ \dfrac{x-b}{c-b} & \text{if } x \in (b,c) \end{cases}$$



Now, if we define

$$V_1 = \{v, v|_{(a,b)} \in P_1(a,b),\ v(x) = 0\ \forall x \notin (a,b)\}$$
$$V_2 = \{v, v|_{(b,c)} \in P_1(b,c),\ v(x) = 0\ \forall x \notin (b,c)\}$$

which are the extensions by zero of the space P_1 we have defined at the beginning of the section, we can also define the space P_1^{disc} as their direct sum, i.e.

$$P_1^{\text{disc}} = V_1 \oplus V_2 = \{v,\ v = v_1 + v_2,\ v_i \in V_i\}$$

This already illustrates the importance of working on individual elements to further define more general spaces.

Now, let us try to find an approximation u_h of u from this space to solve our model problem when $\theta = 0$, but, first of all, recall the exact problem we are dealing with: "Determine $u \in V = H^1(0,1)$, such that $u(0) = 0$ and that

$$\int_0^1 u'\, v'\, dx = \int_0^1 f\, v\, dx$$

holds for all $v \in V$ satisfying $v(0) = 0$." In this case we are taking $a = 0,\ c = 1$. Notice that the integral above can be written as

$$\int_0^1 u'\, v'\, dx = \int_0^b u'\, v'\, dx + \int_b^1 u'\, v'\, dx$$

and similarly for the integral in the right hand side. Motivated by this, consider the following Galerkin formulation: "Determine $u_h \in V_h = P_1^{\text{disc}}$, such that $u_h(0) = 0$ and that

$$\int_0^b u_h'\, v_h'\, dx + \int_b^1 u_h'\, v_h'\, dx = \int_0^b f\, v_h\, dx + \int_b^1 f\, v_h\, dx$$

holds for all $v_h \in V_h$ satisfying $v_h(0) = 0$."

The discrete solution we are looking for is $u_h \in V_h$ and can be written as

$$u_h = \sum_{i=1}^{4} U_j \phi^j(x)$$

(i) First, we have to include the boundary condition in the definition of the space, for which we define

$$V_{h0} = \{v \in P_k^{\text{disc}}, v(0) = 0\}$$

Notice that this removes one degree of freedom, so this subspace has dimension 3 and it is spanned by $\{\phi^2, \phi^3, \phi^4\}$. This is like taking $U_1 = 0$ above.

(ii) Second, we have to compute the coefficients $a_d(\phi^i, \phi^j)$ of the matrix $\underline{\underline{K}} \in \mathbb{R}^{3\times 3}$ appearing in the linear system

$$\underline{\underline{K}}\,\underline{\mathrm{U}} = \underline{\mathrm{F}}$$

Considering the basis of V_{h0} to be the set of functions $\{\psi^1, \psi^2, \psi^3\} = \{\phi^2, \phi^3, \phi^4\}$, we compute the matrix:

$$\underline{\underline{K}} = \begin{bmatrix} a_d(\phi^2,\phi^2) & a_d(\phi^3,\phi^2) & a_d(\phi^4,\phi^2) \\ a_d(\phi^2,\phi^3) & a_d(\phi^3,\phi^3) & a_d(\phi^4,\phi^3) \\ a_d(\phi^2,\phi^4) & a_d(\phi^3,\phi^4) & a_d(\phi^4,\phi^4) \end{bmatrix} \tag{2.14}$$

and calculating the integrals we obtain ...

$$K_{11} = a_d(\phi^2,\phi^2) = \int_0^b (\phi^2)'(\phi^2)'\,dx + \int_b^1 (\phi^2)'(\phi^2)'\,dx = \int_0^b (\phi^2)'(\phi^2)'\,dx + 0 = \frac{1}{b},$$

$$K_{12} = K_{21} = a_d(\phi^3,\phi^2) = \int_0^b (\phi^3)'(\phi^2)'\,dx + \int_b^1 (\phi^3)'(\phi^2)'\,dx = \int_0^b 0\,(\phi^2)'\,dx + \int_b^1 (\phi^3)'\,0\,dx = 0$$

$$K_{13} = K_{31} = a_d(\phi^4,\phi^2) = \int_0^b (\phi^4)'(\phi^2)'\,dx + \int_b^1 (\phi^4)'(\phi^2)'\,dx = \int_0^b 0\,(\phi^2)'\,dx + \int_b^1 (\phi^4)'\,0\,dx = 0$$

... and so on, giving

$$\underline{\underline{K}} = \begin{bmatrix} \frac{1}{b} & 0 & 0 \\ 0 & \frac{1}{1-b} & -\frac{1}{1-b} \\ 0 & -\frac{1}{1-b} & \frac{1}{1-b} \end{bmatrix} \tag{2.15}$$

Notice that the term $K_{11} = a_d(\phi^2, \phi^2)$ is exactly what we had computed before when introducing the $P_1(a, b)$ space simply with $a = 0$, so, we could just have reused that result. Similarly for the second diagonal 2×2 block of (2.15),

$$\begin{bmatrix} \frac{1}{1-b} & -\frac{1}{1-b} \\ -\frac{1}{1-b} & \frac{1}{1-b} \end{bmatrix}$$

which is exactly the matrix we have computed before but in the interval (b, c) instead of (a, b) and taking $c = 1$.

Finally, notice also that $\underline{\underline{K}}$ **is singular!**

- Why did it fail?
- Is the space that we used a subset of $H^1(0, 1)$?

Answer: Functions in this space are discontinuous at $x = b$, therefore this space is not in $H^1(0, 1)$. Actually we have the following important theorem:

Theorem 2.1 *Let v be a* **piecewise-polynomial** *function on a partition of a domain Ω, then*

$$v \in H^1(\Omega) \iff v \in C^0(\bar{\Omega})$$

A more general version of the theorem as well as a proof for the 2D case and P_1 elements will be given later.

In the previous example, since functions in the space are discontinuous, their derivatives appearing in the integrals are Dirac delta functions at $x = b$, so, the integrals are not defined, however, since we partitioned the integrals, we naively proceed with the calculations and obtained a singular matrix. Following with this naive approach, it is interesting also to perform the computation of the system matrix when $\theta = 1$ and see what happens, for which it only remains the computation of matrix $\underline{\underline{M}}$

$$M_{ij} = \int_0^1 \phi^i \, \phi^j \, dx = \int_0^b \phi^i \, \phi^j \, dx + \int_b^1 \phi^i \, \phi^j \, dx$$

Again, we can reuse the results already obtained when describing the space P_1 for a single interval. The final matrix will be the sum of the previously computed $\underline{\underline{K}}$ and $\underline{\underline{M}}$.

$$\underline{\underline{A}} = \underline{\underline{K}} + \underline{\underline{M}} = \begin{bmatrix} \frac{1}{b} + \frac{b}{3} & 0 & 0 \\ 0 & \frac{1}{1-b} + \frac{1-b}{3} & -\frac{1}{1-b} + \frac{1-b}{6} \\ 0 & -\frac{1}{1-b} + \frac{1-b}{6} & \frac{1}{1-b} + \frac{1-b}{3} \end{bmatrix} \tag{2.16}$$

In this case, the matrix is not singular. For instance, if we take the function in the right hand side of the variational formulation to be the constant function $f = 1$ and we calculate the coefficients $\ell(\phi^i)$ of vector

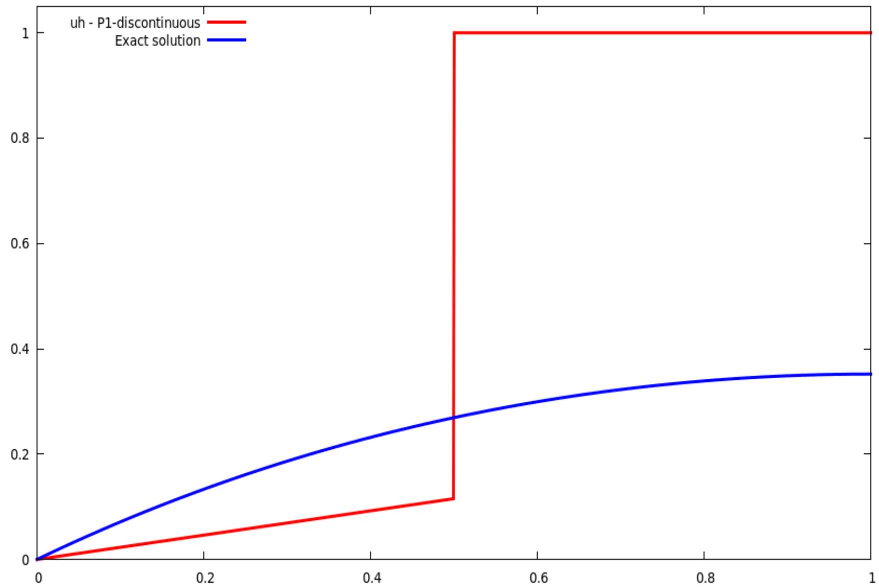
$\underline{\mathrm{F}} \in \mathbb{R}^3$ we get

$$F_1 = \ell(\phi^2) = \int_0^1 \phi^2 \, dx = \frac{b}{2}$$
$$F_2 = \ell(\phi^3) = \int_0^1 \phi^3 \, dx = \frac{1-b}{2}$$
$$F_3 = \ell(\phi^4) = \int_0^1 \phi^4 \, dx = \frac{1-b}{2}$$

Taking now e.g. $b = 0.5$ and finally solving $\underline{\underline{A}}\,\underline{U} = \underline{F}$ we obtain $\underline{U} = \begin{bmatrix} 0.1154 & 1 & 1 \end{bmatrix}^T \Rightarrow u_h = 0.1154\,\phi^2(x) + \phi^3(x) + \phi^4(x)$, which is plotted below and compared with the exact solution for this problem.

This last example serves to illustrate that even when we are able to obtain some result, the approximation we are obtaining lacks of meaning as a consequence of an incorrect choice of the discrete space considered to solve the problem.

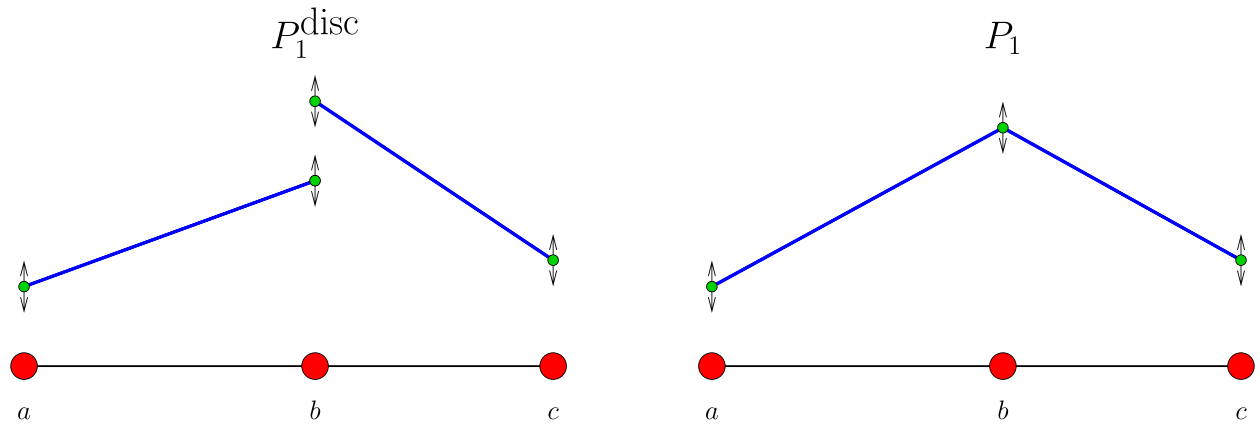
In the next section we remedy this by defining a space of continuous functions.



2.2.3 A P_1 continuous space

Consider the intervals (a, b) and (b, c). In the previous example "glue" the degree of freedom at $x = b$, of the interval to the left and to the right of this point, by imposing the restriction $v(b^-) = v(b^+)$. In this case we only have three degrees of freedom:

$$\begin{aligned}\sigma_1(v) &= v(a)\\ \sigma_2(v) &= v(b)\\ \sigma_4(v) &= v(c)\end{aligned}$$



therefore the space has dimension 3. This choice automatically leads to a space of continuous functions in (a, c) which we describe as

$$V_h = \{v,\ v|_{(a,b)} \in P_1(a,b),\quad v|_{(b,c)} \in P_1(b,c)\}\ \cap\ C^0(a,c)$$

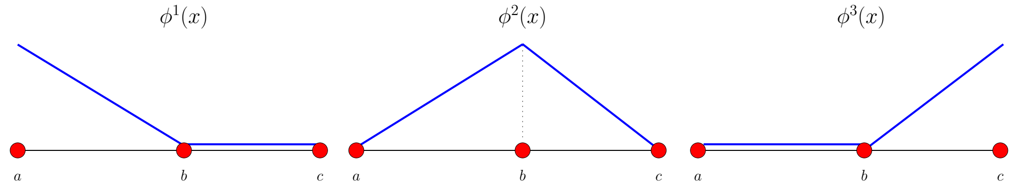
Again, considering

$$v(x) = \begin{cases} \alpha + \beta\, x & \text{if } x \in (a,b) \\ \gamma + \epsilon\, x & \text{if } x \in (b,c) \end{cases}$$

By inspection we find that the basis is:

$$\phi^1(x) = \begin{cases} \dfrac{b-x}{b-a} & \text{if } x \in (a,b) \\ \\ 0 & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^2(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in (a,b) \\ \\ \dfrac{c-x}{c-b} & \text{if } x \in (b,c) \end{cases} \quad , \quad \phi^3(x) = \begin{cases} 0 & \text{if } x \in (a,b) \\ \\ \dfrac{x-b}{c-b} & \text{if } x \in (b,c) \end{cases}$$

which clearly satisfies that $\sigma_i(\phi^j) = \delta_{ij}$.



Exo. 2.3 *Compute the matrix $\underline{\underline{A}}$ for the model problem in this case and compare it with the one obtained when using the P_1^{disc} space. By computing the solution you will notice how good the approximation from this space is as illustrated below.*

Now, we generalize this to partitions of the interval with an increasing number of subintervals:

2.3 1D finite element meshes

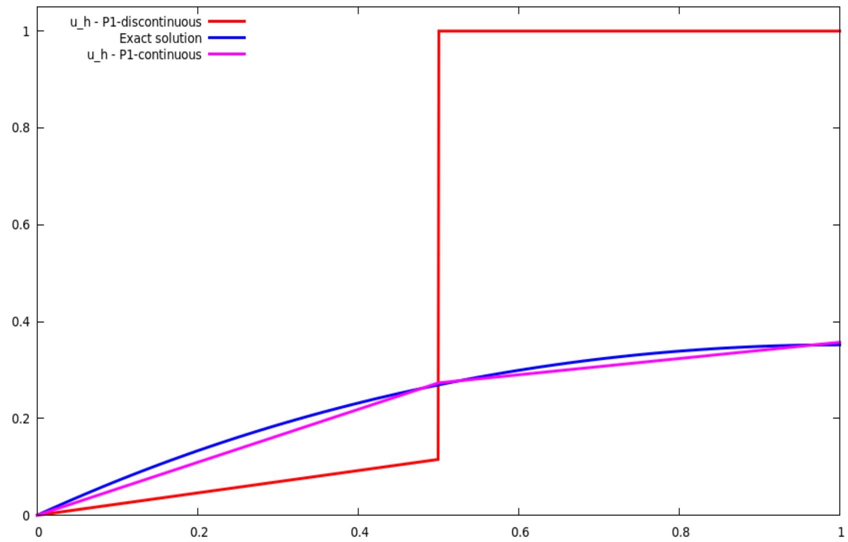
Let consider a partition $\mathcal{T}_h$ of $\Omega = [0, 1]$, i.e., an indexed collection of intervals

$$\bar{\Omega} = \bigcup_{j=1}^{N} I_j$$

where $I_j = [x^j, x^{j+1}]$ and the N_v **nodes** (arbitrarily numbered) are $0 = x^0 < x^1 < x^2 < \cdots < x^N < x^{N+1} = 1$. Define $h_i = x^{i+1} - x^i$ and

$$h = \max_j h_j$$

which is a measure of how fine the partition is.



2.3.1 A $P_1^{\mathbf{disc}}(\mathcal{T}_h)$ (totally discontinuous) space in 1D

With the partition of Ω just defined, we begin by defining the spaces:

$$V_i = \{v, v|_{I_i} \in P_1(I_i),\ v(x) = 0\ \forall x \notin I_i\}$$

where $P_1(I_i) = P_1(x_i, x_{i+1})$ is the space P_1 for an individual interval the we introduced before.

Now, we define a totally discontinuous space associated to the partition $\mathcal{T}_h$ as the direct sum of these V_i's:

$$X_h(\mathcal{T}_h) = V_1 \oplus V_2 + \cdots \oplus V_N = \{v,\ v = v_1 + v_2 + \cdots + v_N,\ v_i \in V_i\}$$

This space has dimension equal to $N \times 2$, but it is not in $H^1(0, 1)$.

Exo. 2.4 *Which degrees of freedom can be chosen in this case?*

2.3.2 $P_1(\mathcal{T}_h)$ conforming space in 1D

Now, if we "glue" the local degrees of freedom of the individual intervals at the corresponding common nodes of $\mathcal{T}_h$, which is equivalent to choosing as degrees of freedom **the values of the function at these nodes**, we naturally define a space of continuous functions

$$V_h = P_1(\mathcal{T}_h) = X_h(\mathcal{T}_h) \ \cap \ C^0(0,1)$$

and the basis functions will be

$$\phi^i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{if } x \in I_{i-1} \\[2ex] \dfrac{x_{i+1} - x}{h_i} & \text{if } x \in I_i \\[2ex] 0 & \text{otherwise} \end{cases}$$

- The dimension of V_h is equal to N_v;
- Since the degress of freedom are the values of the function at the nodes of $\mathcal{T}_h$ and the ϕ^i's are linearly independent, any function in V_h is uniquely determined precisely by these values, i.e.

$$v = \sum_{i=0}^{N+1} U^i \, \phi^i(x) = \sum_{i=0}^{N+1} v(x^i) \, \phi^i(x)$$

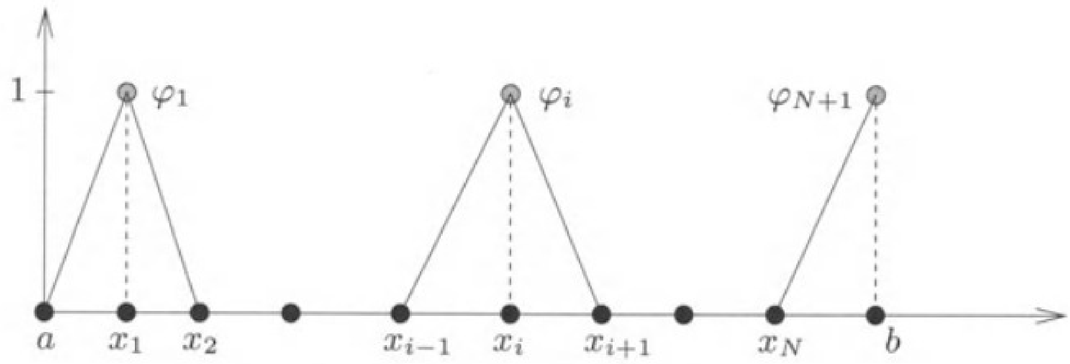
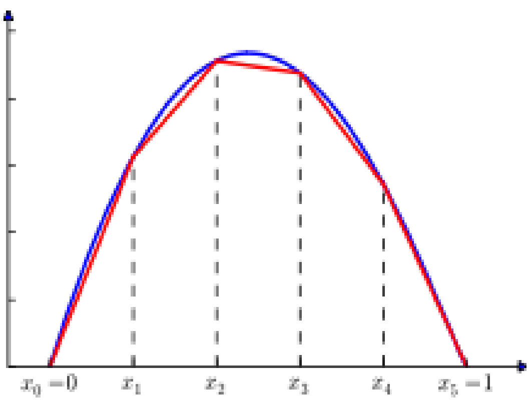


Fig. 1.1. One-dimensional hat functions.



- These functions are linear on each interval (or element) and continuous, but their derivatives are not defined in the classical sense at all points. Is $V_h \subset H^1(0,1)$?

  The answer is YES, as theorem 2.1 states.

We can use this space (introducing first the boundary conditions into its definition) to solve our model problem (see **Exo. 1.16**), and study $\|u - u_h\|_{H^1(0,1)}$ as we refine the partition. One would expect that the Galerkin approximation u_h from this space will converge to the solution u when $h \to 0$, which for this particular case is intuitive, because any continuous function can be approximated by polygonals with an increasing number of nodes.

We will study this in a more general setting in the following sections.

Exo. 2.5 *Do* **Exo. 1.16** *and read Duran's notes!*

2.4 2D examples

2.4.1 P_1 element for a triangle

Consider a triangle K in $\mathbb{R}^2$ with vertices $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$. We want to find a basis for

$$V_h = P_1(K) = \{v : K \to \mathbb{R},\ v = \alpha + \beta x + \gamma\, y\} \tag{2.17}$$

The space has dimension 3.

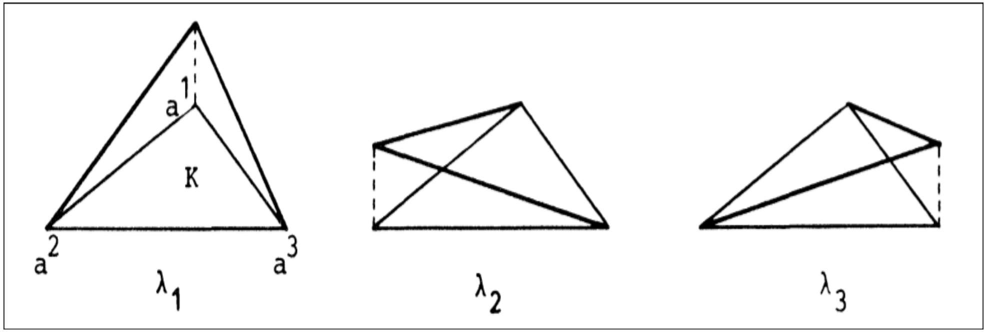
Again we start by defining the degrees of freedom. As done in previous examples we use the value of the function at a set of points, the vertices in this case

$$\sigma_i(v) = v(\mathbf{x}^i) \tag{2.18}$$

and the basis for $P_1(K)$ is defined by the relation $\sigma_i(\phi^j) = \delta_{ij}$. The coefficients of the basis functions are determined by solving the 3×3 systems:

$$\begin{array}{lll}
\alpha_1 + \beta_1\, x^1 + \gamma_1\, y^1 = 1, & \alpha_2 + \beta_2\, x^1 + \gamma_2\, y^1 = 0, & \alpha_3 + \beta_3\, x^1 + \gamma_3\, y^1 = 0 \\
\alpha_1 + \beta_1\, x^2 + \gamma_1\, y^2 = 0, & \alpha_2 + \beta_2\, x^2 + \gamma_2\, y^2 = 1, & \alpha_3 + \beta_3\, x^2 + \gamma_3\, y^2 = 0 \\
\alpha_1 + \beta_1\, x^3 + \gamma_1\, y^3 = 0, & \alpha_2 + \beta_2\, x^3 + \gamma_2\, y^3 = 0, & \alpha_3 + \beta_3\, x^3 + \gamma_3\, y^3 = 1
\end{array}$$

Notice that for simplicity of notation above we have used (x^i, y^i) instead of (x_1^i, x_2^i).



Exo. 2.6 *Write these functions for the master triangle $\hat{K}$ having as vertices* $((0,0),(1,0),(0,1))$.

Exo. 2.7 *Repeat the calculations for quadrilateral elements considering the bilinear functions* $v(x,y) = \alpha_0 + \alpha_1\, x + \alpha_2\, y + \alpha_3\, x\, y$ *and the value of the function at the vertices of the quadrangle as degrees of freedom.*

Exo. 2.8 *The values of the function at the vertices are not the only possible choice as degrees of freedom. Considering as degrees of freedom the line integrals*

$$\sigma_1(v) = \frac{1}{\|\mathbf{x}^2 - \mathbf{x}^3\|} \int_{\mathbf{x}^2}^{\mathbf{x}^3} v(s)\, ds \tag{2.19}$$

$$\sigma_2(v) = \frac{1}{\|\mathbf{x}^3 - \mathbf{x}^1\|} \int_{\mathbf{x}^3}^{\mathbf{x}^1} v(s)\, ds \tag{2.20}$$

$$\sigma_3(v) = \frac{1}{\|\mathbf{x}^1 - \mathbf{x}^2\|} \int_{\mathbf{x}^1}^{\mathbf{x}^2} v(s)\, ds \tag{2.21}$$

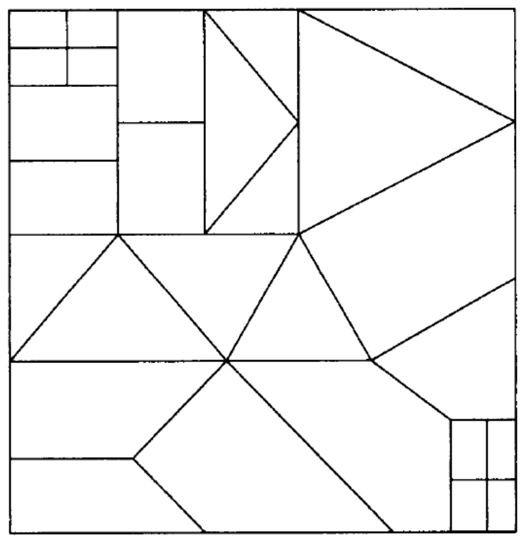
Calculate the basis for a $P_1(K)$-triangle. This is called the Crouzeix-Raviart element.

2.5 2D finite element meshes

Let consider a domain $\Omega \subset \mathbb{R}^2$ and for simplicity assume its boundary $\partial\Omega$ is a polygonal curve. Now, consider a partition $\mathcal{T}_h = \{K_i\}_{i=1}^N$ of Ω, such that

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{K}_i$$

where $K_i \cap K_j = \emptyset$ if $i \neq j$. $\mathcal{T}_h$ is called a triangulation of Ω.

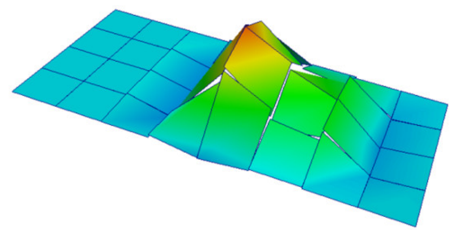


Given a triangulation like those shown, which types of spaces V_h can be constructed?

- We can construct spaces of **discontinuous functions**. If the partition has N_e triangular elements and we consider P_1-triangles for instance, we will have 3 (**local**) degrees of freedom per single triangle. Then a space of totally discontinuous functions associated to the partition $\mathcal{T}_h$ will be the direct sum of (local) P_1 spaces $V_K = \{v : K \to \mathbb{R}, v|_K \in P_1(K),\ v(\mathbf{x}) = 0\ \forall \mathbf{x} \notin K\},\quad K = 1, \ldots, N_e$, i.e.,

$$X_h(\mathcal{T}_h) = V_1 \oplus V_2 + \cdots \oplus V_N = \{v,\ v = v_1 + v_2 + \cdots + v_N,\ v_i \in V_i\}$$

  and its dimension will be $N_e \times 3$, but, remember that this space will not be in $H^1(\Omega)$ (see appendix).



- Also, we can construct spaces of **continuous functions**, but, it happens that this is not trivial in general for the so called nonconforming meshes, for which we have the following definition:

Def. 2.2 *A partition $\mathcal{T}_h$ of a domain Ω is* **conforming** *if $\bar{K}_i \cap \bar{K}_j$ is either*

- *empty, or,*
- *a vertex, or*

- *a complete edge.*

otherwise the partition is said to be **nonconforming**

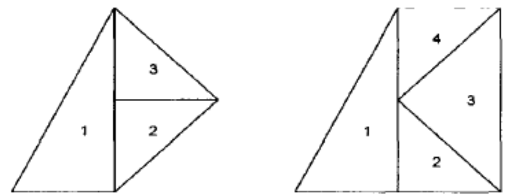


Figure 4.1. *Two examples of nonconforming triangulations. In both examples, the intersection of triangles 1 and 2 is a line segment that is not an edge of triangle 1.*

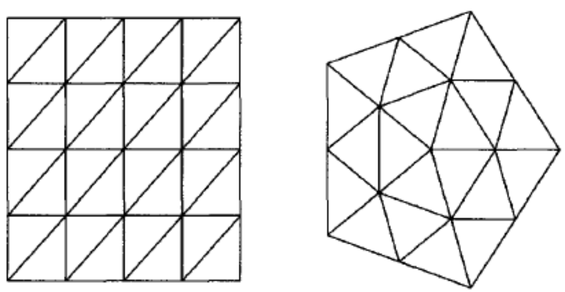


Figure 4.2. *Triangulations of two polygonal domains.*

2.5.1 $P_1(\mathcal{T}_h)$ conforming space in 2D

We proceed similarly to the 1D case. Given a **conforming triangulation** $\mathcal{T}_h$ of a polygonal domain we can build a space of continuous functions. Start with the space

$$X(\mathcal{T}_h) = \{v,\ v|_{K_i} \in P_1(K_i)\ \forall K_i \in \mathcal{T}_h\} \tag{2.22}$$

where $v|_K$ denotes the restriction of v to K and $P_1(K)$ is the space of polynomial functions of degree ≤ 1 on triangle K that we have already defined in subsection 2.4.1

We define as degrees of freedom the value of the function at the nodes of the triangulation.

Since we are assuming now that $\mathcal{T}_h$ is conforming, each vertex of any triangle can only be a vertex of other triangles an cannot be on an edge. Thus, we can "glue" the (local) degrees of freedom of the individual triangles. This naturally leads to the following description of the space we have constructed

$$V_h = X(\mathcal{T}_h)\ \cap\ C^0(\bar{\Omega}) = \{v \in C^0(\bar{\Omega}),\ v|_K \in P_1(K)\ \ \forall K \in \mathcal{T}_h\}$$

We can construct a basis for this space immediately. Let us assume $\mathcal{T}_h$ has N_v vertices whose coordinates are $\{\mathbf{x}^i\}_{i=1}^{N_v}$. Let $\phi^i,\ i = 1, \ldots, N_v$ be the functions that satisfies

$$\phi^i(\mathbf{x}^j) = \delta_{ij} \tag{2.23}$$

whose restriction to element K having j as one of its vertices is the corresponding function in $P_1(K)$ and 0 otherwise. Any function $v = \sum_{i=1}^{N_v} v(\mathbf{x}^i)\,\phi^i(x) \in V_h$ is uniquely determined by the degrees of freedom that are precisely the values of the function at the N_v nodes of $\mathcal{T}_h$. Notice that

- $\{\phi^j\}_{j=1}^{N_v}$ are linearly independent;

- $V_h = \mathrm{span}\{\phi^j\} = \{v_h, v_h = \sum_{j=1}^{N_v} a^j\, \phi^j\}$;
- $\dim(V_h) = N_v$.

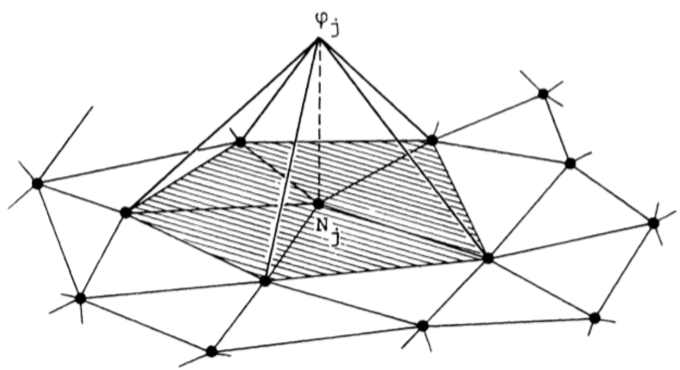
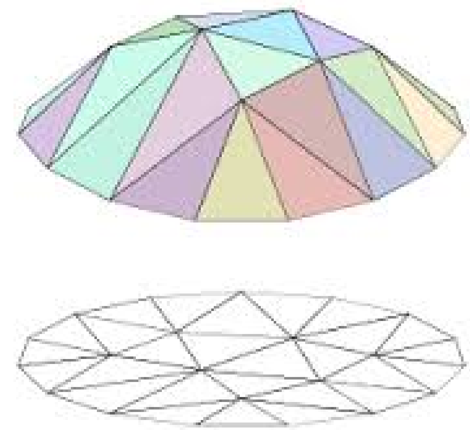


Fig 1.9 The basis function φ_j.



Exo. 2.9 *Show that functions of V_h are actually continuous at the common edge between two triangles of $\mathcal{T}_h$.*

Exo. 2.10 *Noticing that the support of function ϕ^j are all the elements sharing node j, what are the consequences for the matrix $\underline{\underline{A}}$ ($A_{ij} = a(\phi^i, \phi^j)$), when choosing such space to compute an approximation to u?*