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Finite element methods for the Stokes system with interface pressure discontinuities

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Abstract

Surface tension in multi-phase fluid flow engenders pressure discontinuities on phase interfaces. In this work we present two finite element methods to solve viscous incompressible flows problems, especially designed to cope with such a situation. Taking as a model the two-dimensional Stokes system, we consider solution methods based on piecewise linear approximations of both the velocity and pressure, with either velocity bubble or penalty enrichment, in order to obtain stable discrete problems. Moreover a suitable modification of the pressure space is employed in order to represent interface discontinuities. A priori error analyses point to optimal convergence rates for both approaches, which justify observations from previous numerical experiments carried out in [3].

Key words: Embedded discontinuities, Bubble function, finite elements, Galerkin method, Penalty method, interfacial flows, Stokes system, piecewise linear approximation.

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1 Introduction

This work addresses the finite element solution of the following problem describing the flow of a viscous incompressible fluid with viscosity μ occupying two complementary sub-domains Ω_1 and Ω_2 of a bounded domain $\Omega \in \Re^2$ with boundary Γ, separated by an interface Σ. This is assumed to be either an open polygonal line or a Dini-smooth open arc (see e.g. [21]) intersecting Γ at exactly two distinct points in both cases, or yet either a closed Jordan polygonal line or a Dini-smooth Jordan curve [21] completely immersed in Ω. Moreover we assume that Ω is either a polygon or a convex domain such that Γ is of the piecewise C^1 class.

Let $\mathbf{f}$ be a given force field in $\mathbf{L}^2(\Omega)$, and φ be a given force distribution on Σ belonging to $H^{1/2}(\Sigma)$ (cf. [1]). Prescribing a velocity $\mathbf{g} \in \mathbf{C}^0(\Gamma)$ satisfying $\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit outer normal vector on Γ, we wish to find a velocity $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and a pressure $p \in H^1(\Omega\setminus\Sigma)\cap L^2_0(\Omega)$ (cf. [17]) such that:

$$\begin{cases} -\mu\Delta\mathbf{u} + \mathbf{grad}\, p = \mathbf{f} & \text{in } \Omega\setminus\Sigma \\ div\, \mathbf{u} = 0 & \text{in } \Omega \\ [\![p]\!] = \varphi & \text{on } \Sigma \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{1}$$

where $[\![q]\!] := q_{/\partial\Omega_1} - q_{/\partial\Omega_2}$ represents the jump across Σ of the traces of a function $q \in H^1(\Omega\setminus\Sigma)$ on both sides of Σ, $\partial\Omega_i$ being the boundary of Ω_i, $i = 1,2$.

Denoting by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ in scalar, vector or tensor version, with associated norm $\|\cdot\|$, and by $(\cdot,\cdot)_D$ the standard inner product of $L^2(D)$ with associated norm $\|\cdot\|_D$, for any $D \subsetneq \Omega$, we may rewrite problem (1) in the following equivalent variational formulation of the Galerkin type:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H}^1(\Omega) \text{ and } p \in L^2_0(\Omega) & \text{such that} \\ \mu(\mathbf{grad}\,\mathbf{u}, \mathbf{grad}\,\mathbf{v}) - (p, div\,\mathbf{v}) = (\mathbf{f},\mathbf{v}) + (\varphi, \mathbf{v}\cdot\vec{\nu})_\Sigma & \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) \\ (div\,\mathbf{u}, q) = 0 & \forall q \in L^2_0(\Omega) \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\vec{\nu}$ denotes the unit normal vector on Σ oriented from Ω_1 towards Ω_2, and the symbol $\cdot$ is used to represent the euclidean inner product of two vectors of $\Re^N$.

Clearly enough, as a part of the solution of (2), the pressure p is a priori discontinuous on Σ. Therefore it seems natural to approximate such a problem using a pressure finite element space consisting of discontinuous functions on Σ, by matching the mesh in such a way that Σ is approximated by the union of edges of a patch of elements. However in doing so one may be loosing the possibility to work with structured or even uniform meshes, if Ω is rectangular for instance, since Σ may be of very irregular shape. Therefore our numerical approach will be based on a modification of the usual continuous piecewise linear finite element space to represent the pressure, in order to accommodate discontinuities over Σ in the way proposed in [3]. For relevant applications of this method we refer to [8].

Let $\mathcal{P}$ be a quasi-uniform family of partitions $\mathcal{T}_h$ of Ω consisting of straight edge triangles with maximum edge length equal to h, and whose union is a polygonal approximation Ω_h of Ω, in such a way that all of its vertices lie on Γ. We denote by Γ_h the boundary of Ω_h which coincides with Γ in case Ω is a polygon. The assumption that Ω be a convex domain in case it is not a polygon was made here in order to avoid some non essential complications related to external approximations. Indeed if Ω is convex Ω_h is a subset of it for every $\mathcal{T}_h$, and hence we can work with internal approximations only.

Henceforth we denote by $\mathcal{S}_h$ the set of elements in $\mathcal{T}_h$ having a non-empty intersection with Σ. For the sake of simplicity we assume the following mesh configuration: $\forall T \in \mathcal{S}_h$, $T \cap \Sigma$ is a curved or straight segment intersecting the boundary of T at exactly two points different from its vertices; furthermore every vertex of an element in $\mathcal{S}_h$ is also a vertex of at least one element in $\mathcal{T}_h \setminus \mathcal{S}_h$. Notice that both assumptions are very reasonable if h is small enough. The pressure space associated with $\mathcal{T}_h$ of discontinuous functions in the elements of $\mathcal{S}_h$ as described in [3], is denoted here by $\tilde{Q}_h$, and its intersection with $L^2_0(\Omega_h)$ by Q_h. Like in that work this space is defined in connection with the polygonal approximation Σ_h of Σ formed by the chords joining the two end points of $T \cap \Sigma$ for $T \in \mathcal{S}_h$. The subset of Ω_h corresponding to Ω_i out of the two ones separated by Σ_h is denoted by Ω_{hi}, and the unit vector normal to Σ_h oriented from Ω_{h1} towards Ω_{h2} is represented hereafter by $\vec{\nu}_h$.

In this work we use the following notations: the letter C combined or not with other symbols but h represents constants independent of h. $\| \cdot \|_{r,D}$ is the standard norm of Sobolev space $H^r(D)$ for $r \in \Re$, D being a subset of Ω. $| \cdot |_{m,D}$ represents the standard semi-norm of Sobolev space $H^m(D)$, for $m \in \mathbb{N}$, and for $p \in [1, \infty] \setminus \{2\}$, $\| \cdot \|_{m,p,D}$ denotes the standard norm of Sobolev space $W^{m,p}(D)$, D being any open subset of Ω. In all cases we drop the subscript D whenever D is Ω itself. According to the context, $|D|$ or $|\mathbf{d}|$, will denote either the measure of a set $D \subset \Re^N$ or the euclidean norm of a vector $\mathbf{d} \in \Re^N$.

In the sequel $\mathbf{g}_h$ stands for the piecewise linear interpolate of $\mathbf{g}$ along Γ_h at the vertices of $\mathcal{T}_h$ lying on Γ. Moreover we denote by $(\cdot, \cdot)_h$ the standard inner product of $L^2(\Omega_h)$ and by $\| \cdot \|_h$ the associated norm.

2 Finite element solution based on bubble enrichment

In this section we study a first solution method of (1), which is the counterpart of the celebrated Arnold-Brezzi-Fortin method [2], obtained by replacing the classical continuous piecewise linear pressure space with $\tilde{Q}_h$. The bubble enriched space exploited in several contributions to approximate the behavior of incompressible media (see e.g. [10], and [2] itself), is also slightly simplified here. First we define for $i = 1, 2$:

$$\tilde{\Omega}_{hi} = \cup_{T \in \mathcal{T}_h,\, \bar{T} \subset \Omega_i} \{T\}. \tag{3}$$

Let $\mathbf{V}_h$ be the standard space of continuous piecewise linear vector fields associated with $\mathcal{T}_h$ and $\mathbf{V}_h^+$ be the direct sum of $\mathbf{V}_h$ with the space of vector fields $\mathbf{V}_h^B$ spanned by the cubic bubble functions (cf. [10]) of all the elements of $\mathcal{T}_h$, except those in $\mathcal{S}_h$. Let $\mathbf{V}_{h0} := \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega_h)$ and $\mathbf{V}_{h0}^+ := \mathbf{V}_h^+ \cap \mathbf{H}_0^1(\Omega_h)$. Hereafter $\mathbf{u}_{gh} \in \mathbf{V}_h^+$ represents a field satisfying $\mathbf{u}_{gh} = \mathbf{g}_h$ on Γ_h and otherwise arbitrary.
Setting $\mathbf{u}_h^+ = \mathbf{u}_{gh} + \mathbf{u}_h^0$ we consider the following problem to approximate (2):

$$\begin{cases} \text{Find } \mathbf{u}_h^0 \in \mathbf{V}_{h0}^+ \text{ and } p_h^+ \in Q_h \text{such that, } \forall \mathbf{v} \in \mathbf{V}_{h0}^+ \text{ and } \forall q \in Q_h, \\ \mu(\mathbf{grad}\ \mathbf{u}_h^0, \mathbf{grad}\ \mathbf{v})_h - (p_h^+, div\ \mathbf{v})_h = (\mathbf{f}, \mathbf{v})_h - \mu(\mathbf{grad}\ \mathbf{u}_{gh}, \mathbf{grad}\ \mathbf{v})_h + (\varphi, \mathbf{v} \cdot \vec{\nu})_\Sigma \\ (div\ \mathbf{u}_h^0, q)_h = -(div\ \mathbf{u}_{gh}, q)_h \end{cases} \tag{4}$$

Before proceeding to the numerical analysis of problem (4), we give the following technical lemma.

Lemma 2.1 *Let w be a function in $H^k(\Omega)$ for a certain integer $k \geq 1$, and $\Pi_h w$ be its interpolate in a finite dimensional space W_h consisting of continuous functions in every element of $\mathcal{T}_h \subset \mathcal{P}$, having the following property for any non negative integer $l \leq k$:*

$$\left[\sum_{T \in \mathcal{T}_h} \| w - \Pi_h w \|_{l,T}^2 \right]^{1/2} \leq \c{C} h^{k-l} \| w \|_k . \tag{5}$$

Then denoting the trace on Σ_h of a function $\chi \in H^1(\Omega)$ by $\sigma[\chi]$, for a certain constant C_Σ it holds:

$$\| \sigma[w - \Pi_h w] \|_{\Sigma_h} \leq C_\Sigma h^{k-1/2} \| w \|_k \tag{6}$$

PROOF. First we note that $\parallel \sigma[w - \Pi_h w] \parallel^2_{\Sigma_h} = \sum_{S \in \mathcal{S}_h} \parallel \sigma[w - \Pi_h w]_{/S} \parallel^2_{S \cap \Sigma_h}$.
Now we set $g_S = (w - \Pi_h w)_{/S}$ and let $\hat{S}$ be a fixed unit reference element. We further denote by $\mathcal{F}_S$ the affine linear mapping from $S \in \mathcal{S}_h$ onto $\hat{S}$, by $\hat{g}$ the transform of g_S in $\hat{S}$ under $\mathcal{F}_S$, and by $\hat{\Sigma}_S$ the image of $\Sigma_h \cap S$ in $\hat{S}$. In this manner have:

$$\parallel \sigma[g_S] \parallel_{S \cap \Sigma_h} \leq C_\sigma h_S^{1/2} \parallel \hat{g} \parallel_{\hat{\Sigma}_S} . \tag{7}$$

where h_S denotes the largest edge of S. Now using the classical Trace Theorem in $\hat{S}$ (cf. [1]) we obtain:

$$\parallel \hat{g} \parallel_{\hat{\Sigma}_S} \leq \hat{C}_\sigma \parallel \hat{g} \parallel_{1,\hat{S}} . \tag{8}$$

Transforming back to element S, by classical estimates we obtain:

$$\parallel \hat{g} \parallel_{1,\hat{S}} \leq C_S [h_S^{-1} \parallel g_S \parallel_S + |g_S|_{1,S}]. \tag{9}$$

Combining (7), (8), (9), recalling that $g_S = (w - \Pi_h w)_{/S}$, and using (5) with $l = 0, 1$, the result follows. ■

To begin with the analysis of problem (4) we prove that it is well-posed.

Proposition 2.1 *Provided h is sufficiently small problem (4) has a unique solution.*

PROOF. It is well-known that (4) has a unique solution if and only if the following condition holds:

$$\exists \beta_h > 0 \text{ such that } \forall q \in Q_h \sup_{\mathbf{v} \in \mathbf{V}^+_{h0},\, \mathbf{v} \neq \mathbf{0}} \frac{(q, div\, \mathbf{v})_h}{\parallel \mathbf{grad\ v} \parallel_h} \geq \beta_h \parallel q \parallel_h . \tag{10}$$

The above assertion is a consequence of a straightforward adaption of the analysis carried out in [2] for solutions $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$, in order extend it to case of non homogeneous velocity boundary conditions.
Let us then prove the validity of (10).
For this purpose we consider a splitting of Q_h into the direct sum $Q_{h1} \oplus Q_{h2} \oplus Q^0_h$, where :

– $Q_{h1} := \{q \ / \ q \in Q_h,\ q = 0 \text{ in } \Omega_{h2}\}$;
– $Q_{h2} := \{q \ / \ q \in Q_h,\ q = 0 \text{ in } \Omega_{h1}\}$;

– $Q_{h0} := \{q \;/\; q \in Q_h,\ q \text{ is constant in } \Omega_{hi},\ i=1,2\}$.

Let also $\tilde{Q}_{hi}$ be the space of functions defined in $\tilde{\Omega}_{hi}$ consisting of restrictions to this set of functions in Q_{hi}, $i=1,2$. Notice that, while on the one hand necessarily every function in Q_{hi} belongs to $L^2_0(\Omega_{hi})$, on the other hand a priori it does not belong to $L^2_0(\tilde{\Omega}_{hi})$. However we can associate with every function $q_i \in Q_{hi}$ a function $\tilde{q}_i \in \tilde{Q}_h$, whose restriction to $\tilde{\Omega}_{hi}$ belongs to $L^2_0(\tilde{\Omega}_{hi})$, by the relations

$$\tilde{q}_i = q_i - |\tilde{\Omega}_{hi}|^{-1} \int_{\tilde{\Omega}_{hi}} q_i \text{ in } \Omega_{hi} \text{ and } \tilde{q}_i = 0 \text{ in } \Omega_h \setminus \Omega_{hi}. \tag{11}$$

Let now $\mathbf{V}^+_{hi0}$ be the subspace of $\mathbf{V}^+_{h0}$ consisting of fields $\mathbf{v}$ that vanish identically in $\Omega_h \setminus \tilde{\Omega}_{hi}$. Notice that necessarily every $\mathbf{v} \in \mathbf{V}^+_{hi0}$ belongs to $\mathbf{H}^1_0(\Omega_h)$. Let then q_i be given in Q_{hi} and $\tilde{q}_i$ be the associated function in $\tilde{Q}_{hi} \cap L^2_0(\Omega_h)$ through relation (11). According to [17], there exist two strictly positive constants $c(\tilde{\Omega}_{hi})$ and $C(\tilde{\Omega}_{hi})$ such that $\exists \mathbf{v}_i \in \mathbf{H}^1_0(\tilde{\Omega}_{hi})$ fulfilling

$$\begin{cases} (div\ \mathbf{v}_i, q)_{\tilde{\Omega}_{hi}} \geq c(\tilde{\Omega}_{hi}) \parallel q \parallel^2_{\tilde{\Omega}_{hi}} \\ \parallel \mathbf{grad}\ \mathbf{v}_i \parallel_{\tilde{\Omega}_{hi}} \leq C(\tilde{\Omega}_{hi}) \parallel q \parallel_{\tilde{\Omega}_{hi}} . \end{cases} \tag{12}$$

In this way, similarly to Arnold-Brezzi-Fortin [2], (12) allows us to assert that there exist suitable constants $\tilde{C}_{hi}$ and $\tilde{c}_{hi}$ depending on $\tilde{\Omega}_{hi}$, such that for certain $\mathbf{v}_{hi} \in \mathbf{V}^+_{hi0}$ it holds that

$$\begin{cases} (\tilde{q}_i, div\ \mathbf{v}_{hi})_{\tilde{\Omega}_{hi}} \geq \tilde{c}_{hi} \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} \\ \parallel \mathbf{grad}\ \mathbf{v}_{hi} \parallel_{\tilde{\Omega}_{hi}} \leq \tilde{C}_{hi} \parallel \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} . \end{cases} \tag{13}$$

Now noticing that from the assumptions on Σ and $\mathcal{P}$ there exist $\bar{C}_i$ and $\underline{C}_i$ such that $|\Omega_{hi} \setminus \tilde{\Omega}_{hi}| \leq \bar{C}_i h$, and $|\tilde{\Omega}_{hi}|^{-1} |\Omega_{hi}|^{1/2} \leq \underline{C}_i$, from (11) we easily obtain:

$$\parallel \tilde{q}_i - q_i \parallel_{\Omega_{hi}} \leq |\Omega_{hi}|^{1/2} |\tilde{\Omega}_{hi}|^{-1} \left| \int_{\Omega_{hi}/\tilde{\Omega}_{hi}} q_i \right| \leq \underline{C}_i \sqrt{\bar{C}_i h} \parallel q_i \parallel_{\Omega_{hi} \setminus \tilde{\Omega}_{hi}} \leq \mathcal{C}_i h^{1/2} \parallel q_i \parallel_{\Omega_{hi}},$$

which trivially implies

$$\begin{cases} \parallel \tilde{q}_i \parallel_{\Omega_{hi}} \geq (1 - \mathcal{C}_i h^{1/2}) \parallel q_i \parallel_{\Omega_{hi}}, \\ \parallel \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} \leq (1 + \mathcal{C}_i h^{1/2}) \parallel q_i \parallel_{\Omega_{hi}} . \end{cases} \tag{14}$$

Next we claim that there exists a constant $\bar{c}_i$ such that

$$\| \tilde{q}_i \|_{\tilde{\Omega}_{hi}} \geq \bar{c}_i \| \tilde{q}_i \|_{\Omega_{hi}} \;\; \forall q_i \in Q_{hi}. \tag{15}$$

In this aim first we note that from the assumptions on $\mathcal{P}$ there exist two strictly positive constants C_m and C_M such that for every pair of triangles of T and $T^{'}$ in $\mathcal{T}_h$ we have $c_m|T| \leq |T^{'}| \leq C_M|T|$. The set $\Omega_{hi} \setminus \tilde{\Omega}_{hi}$ consists of the union of subsets of elements in $\mathcal{T}_h$, generically denoted by K, which are either convex quadrilaterals or triangles (cf. [3]). From the mesh configuration assumed in Section 1, provided h is small enough, each such a quadrilateral shares two vertices with exactly one element of the mesh contained in $\tilde{\Omega}_{hi}$, and each such a triangle shares at least one vertex with one or more elements contained in $\tilde{\Omega}_{hi}$, in both cases for either $i = 1$ or $i = 2$. It is important to note that the edge of K not containing vertices in common with elements in $\tilde{\Omega}_{hi}$ is contained in Σ_h in both configurations of K.
According to [3], in the case of quadrilaterals a function in Q_{hi} varies linearly in each triangle K_1 and K_2 contained in K, constructed by sub-dividing this quadrilateral by means of one of its diagonals. Let K_1 contain the two vertices of K which are ends of the edge e_K in common with a triangle T_K in $\tilde{\Omega}_{hi}$, ξ_1 be the value of $\tilde{q}_i$ at the vertex of K_1 not belonging to K_2, and ξ_2 be the value of $\tilde{q}_i$ at the other end of e_K. By the construction of $\tilde{Q}_h$ (cf. [3]) we have: $\| \tilde{q}_i \|_K^2 = \{[\xi_1^2 + (\xi_1 + \xi_2)^2/2]|K_1| + [\xi_2^2 + (\xi_1 + \xi_2)^2/2]|K_2|\}/3$. On the other hand denoting by ξ_3 the value of $\tilde{q}_i$ at he vertex of T_K opposite to e_K, we have $\| \tilde{q}_i \|_{T_K}^2 = [(\xi_1 + \xi_2)^2 + (\xi_1 + \xi_3)^2 + (\xi_2 + \xi_3)^2]|T_K|/12$. The square root of the expression in brackets above clearly defines a norm of $\overrightarrow{\xi} := (\xi_1, \xi_2, \xi_3) \in \Re^3$, and hence it is bounded below by a non negative number times $|\overrightarrow{\xi}|$. On the other hand it is an easy matter to derive $\| \tilde{q}_i \|_K^2 \leq C_M|T_K||\overrightarrow{\xi}|^2$. This is sufficient to establish that there exists a constant C_A such that $\forall q_i \in Q_{hi}$,

$$\| \tilde{q}_i \|_{T_K \cup K}^2 \leq C_A \| \tilde{q}_i \|_{T_K}^2 \;\; \forall K \in \Omega_{hi} \setminus \tilde{\Omega}_{hi}, \tag{16}$$

where $\tilde{q}_i$ is associated with q_i through (11).
An inequality of the same type as (16) can be derived even more easily if K is a triangle, T_K being in this case any triangle contained in $\tilde{\Omega}_{hi}$ sharing a vertex with K. Indeed in this case $\| \tilde{q}_i \|_K^2 = |K|\xi_3^2$, ξ_3 being the constant value of $\tilde{q}_i$ all over K [3].
Clearly enough (16) trivially implies (15). Then, provided $h \leq min\{1, (2C_1)^{-2}, (2C_2)^{-2}\}$, this together with (14) and (13) allows us to conclude that the following conditions hold for $i = 1$ or 2:

$$\begin{cases} \exists C_{hi} \text{ and } c_{hi} > 0 \text{ such that } \forall q_i \in Q_{hi} \; \exists \mathbf{v}_i \in \mathbf{V}_{hi0}^{+} \text{ satisfying} \\ (q_i, div\; \mathbf{v}_i)_h \geq c_{hi} \| q_i \|_h^2 \\ \| \mathbf{grad}\; \mathbf{v}_i \|_h \leq C_{hi} \| q_i \|_h \,. \end{cases} \tag{17}$$

In order to conclude our proof that the existence condition (10) is satisfied in the case under study, we still need a condition similar to (17) for the pair of spaces $(Q_{h0}, \mathbf{V}_{h0})$. Let then q_0 be given in Q_{h0}. According to [17] there exists a constant $\bar{C}_0$ independent of q_0 and $\mathbf{v}_0 \in \mathbf{H}_0^1(\Omega_h)$ such that

$$\begin{cases} div\ \mathbf{v}_0 = q_0 \text{ in } \Omega_h \\ \parallel \mathbf{grad}\ \mathbf{v}_0 \parallel_h \leq \bar{C}_0 \parallel q_0 \parallel_h . \end{cases} \tag{18}$$

Let $\Pi_h \mathbf{v}_0$ be the Scott-Zhang interpolate of $\mathbf{v}_0$ in $\mathbf{V}_{h0}$ [23]. We have

$$(div\ \Pi_h \mathbf{v}_0, q_0)_h = \parallel q_0 \parallel_h^2 + (div[\Pi_h \mathbf{v}_0 - \mathbf{v}_0], q_0)_h. \tag{19}$$

Furthermore, η_i being the value of q_0 in Ω_{hi}, for $i = 1, 2$, we have $(div[\Pi_h \mathbf{v}_0 - \mathbf{v}_0], q_0)_h = ([\Pi_h \mathbf{v}_0 - \mathbf{v}_0] \cdot \overrightarrow{\nu}, \eta_2 - \eta_1)_{\Sigma_h}$. On the other hand $\exists C_\Sigma$ such that $\parallel q_0 \parallel_h = [\eta_1^2 |\Omega_{h1}| + \eta_2^2 |\Omega_{h2}|]^{1/2} \geq |\eta_2 - \eta_1| |\Sigma_h|^{1/2} / C_\Sigma$. Thus we have

$$(div[\Pi_h \mathbf{v}_0 - \mathbf{v}_0], q_0)_h \leq C_\Sigma \parallel [\Pi_h \mathbf{v}_0 - \mathbf{v}_0] \cdot \overrightarrow{\nu}_h \parallel_{\Sigma_h} \parallel q_0 \parallel_h . \tag{20}$$

Now according to [14] such operator Π_h satisfies the assumption of Lemma 2.1. Hence

$$\begin{cases} \parallel [\Pi_h \mathbf{v}_0 - \mathbf{v}_0] \cdot \overrightarrow{\nu}_h \parallel_{\Sigma_h} \leq \tilde{C}_0 h^{1/2} \parallel \mathbf{grad}\ \mathbf{v}_0 \parallel_h \\ \parallel \mathbf{grad}\ \Pi_h \mathbf{v}_0 \parallel_h \leq \bar{C}_0 \parallel \mathbf{grad}\ \mathbf{v}_0 \parallel_h . \end{cases} \tag{21}$$

Therefore, combining (19), (20) and (21), provided h is sufficiently small, we can state that:

$$\begin{cases} \exists C_0 \text{ and } c_0 > 0 \text{ such that } \forall q_0 \in Q_{h0}\ \exists \mathbf{v}_{h0} \in \mathbf{V}_{h0}^+ \text{ satisfying} \\ (q_0, div\ \mathbf{v}_{h0})_h \geq c_0 \parallel q_0 \parallel_h^2 \\ \parallel \mathbf{grad}\ \mathbf{v}_{h0} \parallel_h \leq C_0 \parallel q_0 \parallel_h . \end{cases} \tag{22}$$

Now we observe that the three spaces Q_{h1}, Q_{h2} and Q_{h0} are orthogonal in $L_0^2(\Omega_h)$. Moreover for all pairs $(q; \mathbf{v})$ in the product spaces $Q_{h0} \times \mathbf{V}_{hi}^+$ for $i = 1, 2$, $Q_{h1} \times \mathbf{V}_{h2}^+$ and $Q_{h2} \times \mathbf{V}_{h1}^+$, we have $(q, div\ \mathbf{v})_h = 0$. On the other hand, given $q \in Q_h$, for each one of its components $q_1 \in Q_{h1}$, $q_2 \in Q_{h2}$ and $q_0 \in Q_{h0}$ there exists $\mathbf{v}_{hi} \in \mathbf{V}_h^+$ satisfying (22) for $i = 0$ and (17) for $i = 1, 2$. Now for a non-negative real number θ to be specified below we set

$\mathbf{v}_h = \theta\mathbf{v}_{h0} + \mathbf{v}_{h1} + \mathbf{v}_{h2}$. From (22) and (17) we derive:

$$\begin{cases} (q, div\ \mathbf{v}_h)_h \geq \theta c_0 \parallel q_0 \parallel_h^2 + \sum_{i=1}^{2} c_{hi} \parallel q_i \parallel_h^2 + \theta[(q_1, div\ \mathbf{v}_{h0})_h + (q_2, div\ \mathbf{v}_{h0})_h] \\ \parallel \mathbf{grad}\ \mathbf{v}_h \parallel_h \leq [\theta^2 C_0^2 + C_{h1}^2 + C_{h2}^2]^{1/2} \parallel q \parallel_h . \end{cases} \quad (23)$$

Thus it suffices to choose $\theta = \dfrac{c_0 c_h}{4C_0^2}$ with $c_h = \min[c_{h1}, c_{h2}]$ to have,

$$\begin{cases} (q, div\ \mathbf{v}_h)_h \geq c_{h3} \parallel q \parallel_h^2 \\ \parallel \mathbf{grad}\ \mathbf{v}_h \parallel_h \leq C_{h3} \parallel q \parallel_h . \end{cases} \quad (24)$$

with $c_{h3} = \min\left[\dfrac{c_0^2 c_h}{8C_0^2}, \dfrac{c_h}{2}\right]$ and $C_{h3} = \left\{\left[\dfrac{c_0 c_h}{4C_0}\right]^2 + C_{h1}^2 + C_{h2}^2\right\}^{1/2}$. Then (10) directly follows from (24) with $\beta_h = c_{3h}/C_{3h}$. ■

In order carry out a convergence analysis for our approximate problem (4), we make the following additional assumptions on Ω, Σ and the family of meshes $\mathcal{P}$.

Assumption:* For h smaller than a certain h_0 there exists a finite partition of $\tilde{\Omega}_{hi}$, say $\{\Omega_{hi}^j\}_{j=1}^{J_i}$ consisting of the union of elements in $\mathcal{T}_h$ such that for each j all the members of the family of domains $\{\Omega_{hi}^j\}_h$ are star-shaped with respect to a disk (cf. [19] page 14) D_i^j with radius $r_i^j > 0$ independent of h.

Remark 1 Assumption would be easily satisfied if $\tilde{\Omega}_{hi}$ were Ω_{hi} itself. Thus the difficulty, if any, concerns only neighborhoods of Σ. Put in simple terms Assumption* holds if the interface Σ has a moderate curvature in the neighborhood B of pairs of sub-domains say Ω_{h1}^{j1} and Ω_{h2}^{j2}, and the successive mesh refinements of each one of them attempt to follow this curve in such a way that no angles of the intersection of Σ_h with the edges of the triangles in $\mathcal{S}_h$ in B are too close to either zero or π. As an illustration of such a situation one might consider the case where $\Sigma \cap B$ is a slight perturbation of a straight line separating two subsets of Ω_{h1} and Ω_{h2} forming a certain region of Ω_h, each one of them being a star-shaped domain with respect to a disk and that the elements in such a region belonging to $\mathcal{S}_h$ form a band of elements such that Σ cuts each one of them not so far from two edge mid-points.* ■

Proposition 2.2 *Provided h is sufficiently small, under Assumption* the fol-*

lowing error bound holds:

$$\| \mathbf{grad}(\mathbf{u} - \mathbf{u}_h^+) \|_h + \| p - p^+ \|_h \leq \tilde{C}_B \left[\inf_{\mathbf{v} \in \mathbf{V}_h^+} \| \mathbf{grad}(\mathbf{u} - \mathbf{v}) \|_h + \inf_{q \in \tilde{Q}_h} \| p - q \|_h \right] \quad (25)$$

PROOF. The existence of $\tilde{C}_B$ such that the error bound (25) holds is guaranteed by standard arguments (see e.g. [6], [4] and [25]) provided the constant $\beta_h > 0$ in (10) is independent of h. Referring to the proof of Proposition 2.1 the mesh independence of β_h is a direct a consequence of the fact that the constants $c(\tilde{\Omega}_{hi})$ and $C(\tilde{\Omega}_{hi})$ appearing in (12) can be taken independently of $\tilde{\Omega}_{hi}$, i.e. of h. According to the work of Bogovskii [5] extended and further exploited in [20], [11] and [16] if all the domains in the family $\{\tilde{\Omega}_{hi}\}_h$ are star-shaped with respect to a single disk D_i, $i = 1, 2$, this mesh independence of both constants can be asserted. In case not Assumption* allows us to split the family $\{\tilde{\Omega}_{hi}\}_h$ into J_i families of star-shaped domains $\{\Omega_{hi}^j\}_h$, $j = 1, \ldots, J_i$ with respect to a fixed disk D_i^j with radius $r_i^j > 0$ bounded below away from zero.

Before pursuing we introduce a variant $\tilde{Q}_h^{disc}$ of $\tilde{Q}_h$ consisting of functions which are continuous in every Ω_{hi}^j and linear in each element of $\mathcal{T}_h$ contained in $\bar{\Omega}_{hi}^j$ $\forall j \in \{1, \ldots, J_i\}$, $i = 1, 2$. We also define $Q_h^{disc} := \tilde{Q}_h^{disc} \cap L_0^2(\Omega_h)$. Moreover for convenience we make the easy-to-satisfy assumption that the Ω_{hi}^j's are numbered in such a way that $\bar{\Omega}_{hi}^j \cap \bar{\Omega}_{hi}^{j+1}$ has a non zero measure bounded away from zero by a strictly positive constant independent of h for $j = 1, \ldots, J_i - 1$ and for every h.

Now let Q_{hi}^{disc} be the subspace of Q_h^{disc} consisting of functions in $L_0^2(\Omega_h)$ which vanish identically in $\Omega_h \setminus \tilde{\Omega}_{hi}$, $i = 1, 2$. We further split Q_{hi}^{disc} into the direct sum of orthogonal subspaces all belonging to $L_0^2(\Omega_h)$, say Q_{hi}^j, $j = 1, \ldots, J_i$ in the same manner as Q_{hi} for Ω_{hi}, plus Q_{h0}^j in case $J_i > 1$, for $j = 1, \ldots, J_i - 1$, where a function in Q_{h0}^j is constant in each one of the domains Ω_{hi}^{j+1} and $\cup_{k=1}^j \Omega_{hi}^k$. In this manner, using Bogovskii's results [5] we can apply the same arguments as in the proof of Proposition 2.1, to establish counterparts of (12) with constants c_i^j and C_i^j all independent of h, for $j = 1, \ldots, J_i$, and $i = 1, 2$. Then we employ repeatedly arguments very much similar to those leading to analogs of (23) and (24) with constants bounded away form zero independently of h, by aggregating the domains Ω_{hi}^j step by step until $\Omega_{hi}^{J_i}$. This leads to the conclusion that

$$\begin{cases} \exists c_3 \text{ and } C_3 \text{ such that } \forall q \in Q_h^{disc} \; \exists \mathbf{v}_h \in \mathbf{H}_0^1(\Omega_h) \text{ satisfying} \\ (q, div\, \mathbf{v}_h)_h \geq c_3 \| q \|_h^2 \\ \| \mathbf{grad}\, \mathbf{v}_h \|_h \leq C_3 \| q \|_h . \end{cases} \quad (26)$$

Noticing that $Q_h \subset Q_h^{disc}$ the result follows from (26) with $\beta_h = \beta = c_3 / C_3$. ■

Now letting ϵ be an arbitrary strictly positive number, we recall a result of [7], according to which $\inf_{q \in \tilde{Q}_h} \parallel p - q \parallel_h$ is bounded above by a term of the form $C_\epsilon h^{1+\epsilon} \parallel p \parallel_{1,2+\epsilon,\Omega \setminus \Sigma}$ if $p \in W^{1,2+\epsilon}(\Omega \setminus \Sigma)$. Hence, from standard approximation results, and using Proposition 2.1, the following a priori estimate holds:

Theorem 2.2 *If the solution of (2) is such that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in W^{1,2+\epsilon}(\Omega \setminus \Sigma)$*, then under Assumption* * $\exists C_B$ *such that*

$$\parallel \mathbf{grad}\ (\mathbf{u} - \mathbf{u}_h^+) \parallel_h \ + \ \parallel p - p_h^+ \parallel_h \leq \ C_B h \left[\parallel \mathbf{u} \parallel_2 + \parallel p \parallel_{1,2+\epsilon,\Omega\setminus\Sigma} \right] . \blacksquare \quad (27)$$

3 Finite element solution based on penalty enrichment

In this section we consider a second finite element solution method, which gathers well-known advantages of the strategy of the Petrov-Galerkin type proposed by Hughes, Franca and Balestra [18] and Douglas and Wang [12] to solve the Stokes system (1) with classical continuous piecewise linear elements. This is because on the one hand the corresponding discrete problem is symmetric, and on the other hand its stability is guaranteed independently of unknown mesh-dependent parameters.
Our formulation is based on the addition of a term to the standard Galerkin formulation, that can be viewed as a penalty term of the divergence free condition, with a penalty parameter δ equal to h^2.
Before stating the corresponding approximate problem, we introduce the following concept of piecewise gradient denoted by $\mathbf{grad}_h$, such that $\forall q \in H^1(\Omega \setminus \Sigma)$, $\mathbf{grad}_h\ q = \mathbf{grad}\ q$ everywhere in Ω, except in the elements of $\mathcal{S}_h$, where $\mathbf{grad}_h\ q = \mathbf{0}$.
We will be dealing with the following approximate problem:

$$\begin{cases} \text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ and } p_h \in Q_h \text{ such that } \forall \mathbf{v} \in \mathbf{V}_{h0} \text{ and } \forall q \in Q_h, \\ \mu(\mathbf{grad}\ \mathbf{u}_h, \mathbf{grad}\ \mathbf{v})_h - (p_h, div\ \mathbf{v})_h = (\mathbf{f}, \mathbf{v})_h + (\varphi, \mathbf{v} \cdot \overrightarrow{\nu})_\Sigma \\ -\delta(\mathbf{grad}_h\ p_h, \mathbf{grad}_h\ q)_h \ - \ (div\ \mathbf{u}_h, q)_h = 0 \\ \mathbf{u}_h = \mathbf{g}_h \text{ on } \Gamma_h. \end{cases} \quad (28)$$

The first important issue to be addressed in connection with problem (28) is its well-posedness.

Proposition 3.1 *Problem (28) has a unique solution, provided h is small enough.*

PROOF. Clearly enough (28) is a linear problem equivalent to an $L_h \times L_h$ linear system of algebraic equations where $L_h = dim\mathbf{V}_h + dimQ_h$. Hence it has a unique solution if and only if the underlying homogeneous system admits only the trivial solution. Letting then $\mathbf{f} = \mathbf{0}$ and $\mathbf{g}_h = \mathbf{0}$, setting $(\mathbf{v}; q) = (\mathbf{u}_h; -p_h)$, and adding up both equations of (28) we obtain $\mu \parallel \mathbf{grad}\ \mathbf{u}_h \parallel_h^2 + \delta \parallel \mathbf{grad}_h\ p_h \parallel_h^2 = 0$. This implies that $\mathbf{u}_h \equiv \mathbf{0}$ and p_h is constant in $\tilde{\Omega}_{hi}$, $i = 1, 2$. However thanks to the assumption on the mesh configuration made in Section 1, we can assert that p_h is constant in each Ω_{hi}, and hence $[\![p_h]\!]$ is constant along Σ_h. Notice that from the first equation of (28) we have $([\![p_h]\!], \mathbf{v} \cdot \vec{\nu}_h)_{\Sigma_h} = 0\ \forall \mathbf{v} \in \mathbf{V}_{h0}$. Let us then choose $\mathbf{v}$ equal to $\mathbf{0}$ at every vertex of $\mathcal{T}_h$ but a certain vertex $P \notin \Gamma_h$ common to a few elements of $\mathcal{S}_h$, where $\mathbf{v}(P)$ is taken so as to verify $\int_{\Sigma_h} \mathbf{v} \cdot \vec{\nu}_h ds = [\![p_h]\!]$. This is possible for we have assumed that h is sufficiently small, and therefore the direction of $\vec{\nu}_h$ in these few elements of $\mathcal{S}_h$ surrounding P cannot change so abruptly, to the point of making $\mathbf{v} \cdot \vec{\nu}_{h/\Sigma_h}$ change of sign. Hence $[\![p_h]\!]$ must be zero, which implies in turn that p_h is constant in the whole Ω_h. Since p_h belongs to $L_0^2(\Omega_h)$ it must vanish identically and we are done. ■

Now in order to establish the uniform stability of problem (28), we need the following result analogous to the one proved in [26] for the $P_1 isoP_2$ Taylor-Hood elements.

Proposition 3.2 *Provided h is sufficiently small and Assumption * holds there exist constants c_1, c_2 and C such that for every $q \in Q_h \setminus \{0\}$ it is possible to find $\mathbf{v}_h \in \mathbf{V}_{h0}$ satisfying*

$$\begin{aligned} &(div\ \mathbf{v}_h, q)_h \geq c_1 \parallel q \parallel_h^2 - c_2 h^2 \parallel \mathbf{grad}_h\ q \parallel_h^2 \\ &\parallel \mathbf{grad}\ \mathbf{v}_h \parallel_h \leq C \parallel q \parallel_h . \end{aligned} \tag{29}$$

PROOF. Let us first assume that the splitting in the statement of Assumption * is true for $J_1 = J_2 = 1$. Referring to the proof of Proposition 2.1 we consider the splitting of Q_h into the direct sum $Q_{h1} \oplus Q_{h2} \oplus Q_{h0}$. Let then q be a given non zero function in Q_h and q_j be its components in Q_{hj}, for $j = 0, 1, 2$.
Recalling the definitions of $\tilde{\Omega}_{hi}$ for $i = 1, 2$ given in the proof of Proposition 2.1, we further introduce the spaces $\mathbf{V}_{hi}$ consisting of fields in $\mathbf{H}_0^1(\tilde{\Omega}_{hi})$, which are linear in each element of $\mathcal{T}_h$ contained in $\tilde{\Omega}_{hi}$.
Similarly to the case of Proposition 2.1 we define $\tilde{q}_i$ to be the function in $L_0^2(\tilde{\Omega}_{hi})$ given by $\tilde{q}_i := q_{i/\tilde{\Omega}_{hi}} - |\tilde{\Omega}_{hi}|^{-1} \int_{\tilde{\Omega}_{hi}} q_{i/\tilde{\Omega}_{hi}}$. Next referring to the proof of Proposition 2.2, we know that $\exists \tilde{C}_i$ depending on $\mathcal{P}$, but certainly not on h,

together with $\tilde{\mathbf{v}}_i \in \mathbf{H}_0^1(\tilde{\Omega}_{hi})$ for both $i = 1$ and $i = 2$, satisfying:

$$\begin{cases} (div\ \tilde{\mathbf{v}}_i, \tilde{q}_i)_{\tilde{\Omega}_{hi}} \geq \tilde{c}_i \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} \\ \parallel \mathbf{grad}\ \tilde{\mathbf{v}}_i \parallel_{\tilde{\Omega}_{hi}} \leq \tilde{C}_i \parallel \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} . \end{cases} \tag{30}$$

Now let $\tilde{\mathbf{v}}_{hi}$ be the Scott-Zhang (cf. [23]) interpolate of $\tilde{\mathbf{v}}_i$ in $\mathbf{V}_{hi}$. We know that $\exists \bar{C}_i$ such that

$$\parallel \tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_{hi} \parallel_{\tilde{\Omega}_{hi}} + h \parallel \mathbf{grad}(\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_{hi}) \parallel_{\tilde{\Omega}_{hi}} \leq \bar{C}_i h \parallel \mathbf{grad}\ \tilde{\mathbf{v}}_i \parallel_{\tilde{\Omega}_{hi}} \tag{31}$$

Using (30) and integration by parts in $\tilde{\Omega}_{hi}$ we obtain:

$$(div\ \tilde{\mathbf{v}}_{hi}, \tilde{q}_i)_{\tilde{\Omega}_{hi}} = (div\ \tilde{\mathbf{v}}_i, \tilde{q}_i)_{\tilde{\Omega}_{hi}} + (div[\tilde{\mathbf{v}}_{hi} - \tilde{\mathbf{v}}_i], \tilde{q}_i)_{\tilde{\Omega}_{hi}} \geq \tilde{c}_i \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} + (\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_{hi}, \mathbf{grad}\ \tilde{q}_i)_{\tilde{\Omega}_{hi}} \tag{32}$$

Then from (30), (31) and (32) we easily derive,

$$\begin{aligned} (div\ \tilde{\mathbf{v}}_{hi}, \tilde{q}_i)_{\tilde{\Omega}_{hi}} &\geq \tilde{c}_i \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} - \bar{C}_i h \parallel \mathbf{grad}\ \tilde{\mathbf{v}}_i \parallel_{\tilde{\Omega}_{hi}} \parallel \mathbf{grad}\ \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} \\ &\geq \tilde{c}_i \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} - \tilde{C}_i \bar{C}_i h \parallel \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} \parallel \mathbf{grad}\ \tilde{q}_i \parallel_{\tilde{\Omega}_{hi}} . \end{aligned} \tag{33}$$

which implies

$$(div\ \tilde{\mathbf{v}}_{hi}, \tilde{q}_i)_{\tilde{\Omega}_{hi}} \geq \tilde{c}_i \parallel \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} /2 - [\tilde{C}_i \bar{C}_i h]^2/(2\tilde{c}_i) \parallel \mathbf{grad}\ \tilde{q}_i \parallel^2_{\tilde{\Omega}_{hi}} . \tag{34}$$

Now we extend $\tilde{\mathbf{v}}_{hi}$ by zero in $\Omega_h \setminus \tilde{\Omega}_{hi}$ to the whole Ω_h, thereby constructing a field $\mathbf{v}_{hi} \in \mathbf{V}_h$. Since the difference between $\tilde{q}_i$ and $q_{i_{/\tilde{\Omega}_{hi}}}$ is constant, we have

$$(div\ \tilde{\mathbf{v}}_{hi}, \tilde{q}_i)_{\tilde{\Omega}_{hi}} = (div\ \tilde{\mathbf{v}}_{hi}, q_i)_{\tilde{\Omega}_{hi}} = (div\ \mathbf{v}_{hi}, q_i)_h. \tag{35}$$

Then recalling (14) and (15), from (34) and (35) we easily derive:

$$(div\ \mathbf{v}_{hi}, q_i)_h \geq [(1 - \mathcal{C}_i h^{1/2})\bar{c}_i]^2 \tilde{c}_i \parallel q_i \parallel^2_h /2 - [\tilde{C}_i \bar{C}_i h]^2/(2\tilde{c}_i) \parallel \mathbf{grad}_h\ q_i \parallel^2_h . \tag{36}$$

Next for a suitable $\theta > 0$ that we select in the same manner as in Proposition 2.1, we define the field $\mathbf{v}_h = \theta \mathbf{v}_{h0} + \mathbf{v}_{h1} + \mathbf{v}_{h2}$, where $\mathbf{v}_{h0}$ satisfies (22). Then we add up relations (36) for $i = 1$ and $i = 2$ and the first one of (22) pre-multiplied by θ. Noting that $\parallel \mathbf{grad}_h\ q_0 \parallel_h = 0$, and that $(div\ \mathbf{v}_{hi}, q_j)_h = 0$ for $i \neq j$, $i \in \{1, 2\}$ and $j \in \{0, 1, 2\}$, this readily leads to (29) for suitable strictly positive constants c_1 and c_2.
Finally the case where either J_1 or J_2 is greater than one can be addressed

through further splittings of Q_{hi}^{disc} into the sum of orthogonal subspaces of $L_0^2(\Omega_h)$. Then following the same script as in the proof of Proposition 2.2, we establish step by step (30) for an arbitrary $\tilde{q}_i \in Q_{hi}^{disc}$, with constants $\tilde{c}_i$ and $\tilde{C}_i$ independent of h. From that point (29) follows in the same way as above since $Q_h \subset Q_h^{disc}$. ■

In view of Proposition 3.2 we may proceed basically like in [15]. First we introduce a symmetric bilinear form $a_h : (\mathbf{V}_h \times Q_h) \times (\mathbf{V}_h \times Q_h) \longrightarrow \Re$, associated with problem (28), namely,

$$a_h((\mathbf{u};q),(\mathbf{v};p)) := \mu(\mathbf{grad}\ \mathbf{u},\mathbf{grad}\ \mathbf{v})_h - (q, div\ \mathbf{v})_h - (div\ \mathbf{u}, p)_h - \delta(\mathbf{grad}_h\ q, \mathbf{grad}_h\ p)_h \quad (37)$$

Letting $\mathbf{u}_{gh} \in \mathbf{V}_h$ be any field such that $\mathbf{u}_{gh_{/\Gamma_h}} = \mathbf{g}_h$, problem (28) can be recast in the equivalent form:

$$\begin{cases} \text{Find } (\mathbf{u}_{0h};p_h) \in \mathbf{V}_{h0} \times Q_h \text{ such that} \\ a_h((\mathbf{u}_{0h};p_h),(\mathbf{v};q)) = -a_h((\mathbf{u}_{gh};0),(\mathbf{v};q)) + (\mathbf{f},\mathbf{v})_h + (\varphi, \mathbf{v}\cdot\overrightarrow{\nu})_\Sigma\ \forall(\mathbf{v};q) \in \mathbf{V}_{h0} \times Q_h. \end{cases} \quad (38)$$

We establish in the following Proposition that a_h is weakly coercive on $(\mathbf{V}_{h0} \times Q_h) \times (\mathbf{V}_{h0} \times Q_h)$ (cf. [4]). As a consequence, whatever the choice of $\mathbf{u}_{gh}$, the sum $\mathbf{u}_{0h} + \mathbf{u}_{gh}$ is invariant, and equal to the first component of the solution to (28).

Proposition 3.3 *If δ is chosen equal to h^2, then $\exists \alpha > 0$ independent of h such that a_h given by (37) satisfies:*

$$\begin{cases} \forall(\mathbf{u};q) \neq (\mathbf{0};0) \in \mathbf{V}_{h0} \times Q_h\ \exists(\mathbf{v};p) \in \mathbf{V}_{h0} \times Q_h \textit{ such that} \\ \dfrac{a_h((\mathbf{u};q),(\mathbf{v};p))}{|||(\mathbf{v};p))|||} \geq \alpha|||(\mathbf{u};q)|||, \\ \textit{where } |||(\mathbf{v};p)||| := [\|\ \mathbf{grad}\ \mathbf{v}\ \|_h^2 + \|\ p\ \|_h^2]^{1/2}\ \forall(\mathbf{v};p) \in \mathbf{V}_{h0} \times Q_h. \end{cases} \quad (39)$$

PROOF. Given $(\mathbf{u};q) \in \mathbf{V}_{h0} \times Q_h \neq (\mathbf{0};0)$, it suffices to choose $(\mathbf{v};p) = (\mathbf{u};-q) - \gamma(\mathbf{v}_h;0)$, where $\mathbf{v}_h$ is the field in $\mathbf{V}_{h0}$ satisfying (29) for such a q, and take $\gamma = \min[c_1 C^{-1}, c_2^{-1}]$ to have (39) with $\alpha = \dfrac{\min[\gamma c_1 - \gamma^2 C/2; \mu/2]}{1+\gamma C} > 0$. ■

Since form a_h satisfies (39) problem (28) has a unique solution. Next we establish the following error bound for it using the weak-coerciveness of a_h:

Proposition 3.4 *Let A be the norm of a_h, namely,*

$$A := \sup_{((\mathbf{u};q);(\mathbf{v};p))\in(\mathbf{V}_h\times Q_h)\times(\mathbf{V}_h\times Q_h)\neq((\mathbf{0};0);(\mathbf{0};0))} \frac{a_h((\mathbf{u};q),(\mathbf{v};p))}{|||(\mathbf{u};q))|||\ |||(\mathbf{v};p))|||}$$

.

Then there exists a constant $\tilde{C}$ depending only on α and A such that,

$$|||(\mathbf{u}_h-\mathbf{u};p_h-p)|||\leq \tilde{C}\left[\inf_{(\mathbf{v};q))\in(\mathbf{V}_h\times Q_h)} |||(\mathbf{u}_h-\mathbf{v};p_h-q)|||+\sqrt{\delta}\parallel \mathbf{grad}_h\ p\parallel_h\right] \quad (40)$$

Proof. From (39) we have,

$$\alpha|||(\mathbf{u}_h-\mathbf{u}_{gh}-\mathbf{v};p_h-q)|||\leq \sup_{(\mathbf{w};o)\in\mathbf{V}_{h0}\times Q_h\neq(\mathbf{0};0)} \frac{a_h((\mathbf{u}_h-\mathbf{u}_{gh}-\mathbf{v};p_h-q),(\mathbf{w};o))}{|||(\mathbf{w};o)|||}\ \forall(\mathbf{v};q)\in\mathbf{V}_{h0}\times Q_h. \quad (41)$$

Then following [13] and [22], we derive:

$$\alpha|||(\mathbf{u}_h-\mathbf{u};p_h-p)|||\leq A\inf_{(\mathbf{v};q)\in\mathbf{V}_h\times Q_h}|||(\mathbf{u}-\mathbf{v};p-q)||| + \sup_{(\mathbf{w};o)\in\mathbf{V}_{h0}\times Q_h\neq(\mathbf{0};0)} \frac{|a_h((\mathbf{u};p),(\mathbf{w};o))-(\mathbf{f},\mathbf{w})-(\varphi,\mathbf{w}\cdot\overrightarrow{\nu})_\Sigma|}{|||(\mathbf{w};o)|||} \quad (42)$$

The numerator of the second term on the right hand side of (42) is easily seen to be equal to $\delta|(\mathbf{grad}_h\ p,\mathbf{grad}_h\ o)_h|$. Thus using the classical inverse inequality in $\tilde{\Omega}_{1h}\cup\tilde{\Omega}_{h2}$ [9] for functions in $\tilde{Q}_h$ we have $\parallel \mathbf{grad}_h\ o\parallel_h\leq C_I h^{-1}\parallel o\parallel_h$. Applying this result we obtain,

$$\delta|(\mathbf{grad}_h\ p,\mathbf{grad}_h\ o)_h|\leq C_I\delta h^{-1}\parallel \mathbf{grad}_h\ p\parallel_h |||(\mathbf{w};o)|||. \quad (43)$$

Since $h=\sqrt{\delta}$, combining (42) and (43) the result follows. ■

Proposition 3.4 is all we need to derive a priori error estimates for problem (28) according to

Theorem 3.1 *If $\delta=h^2$, under Assumption* and the same regularity assumptions of Theorem 2.2 for the solution of (1), there exists a constant C_P such that*

$$\parallel \mathbf{grad}\ (\mathbf{u}-\mathbf{u}_h)\parallel_h+\parallel p-p_h\parallel_h\leq C_P h\left[\parallel \mathbf{u}\parallel_2+\parallel p\parallel_{1,2+\epsilon,\Omega\setminus\Sigma}\right]. \quad (44)$$

PROOF. This result is a trivial consequence of Proposition 3.4 and of the fact that the embedding of $W^{1,2+\epsilon}(D)$ in $H^1(D)$ is continuous $\forall D \subset \Omega$. Indeed, it suffices to take in (40), $\mathbf{v}$ equal to the standard $\mathbf{V}_h$-interpolate of $\mathbf{u}$ and q to be the Q_h-interpolate of p [7], and to bound $\| \mathbf{grad}_h\, p \|_h$ by $\| p \|_{1,2+\epsilon,\Omega\setminus\Sigma}$ ■

4 Data approximation

We have already dealt in the two previous sections with the approximation of the boundary datum $\mathbf{g}$, together with Γ itself in a rather standard manner. In this Section we further address this issue with respect to the other data of (1), namely, $\mathbf{f}$, φ and Σ.
As far as $\mathbf{f}$ is concerned this problem is also standard. For instance, assuming that $\mathbf{f} \in \mathbf{W}^{1,p}(\Omega)$ for $p > 2$, we may replace the term $(\mathbf{f}, \mathbf{v})_h$ with $(\mathbf{f}_h, \mathbf{v})_h$ in both (4) and (28), where $\mathbf{f}_h$ is defined by: $\forall T \in \mathcal{T}_h$ $\mathbf{f}_{h/T} = \mathbf{f}(G_T)$, G_T being the barycenter of T. Then using classical techniques to deal with the so-called variational crimes (cf. [24]), the additional error introduced by such an approximation of $\mathbf{f}$ in the estimates (27) and (44) can be bounded by $C_F h \| \mathbf{f} \|_{1,p}$
The case of φ and Σ is non standard and more delicate. Before going into details, we would like to point out that the approximation of Σ is not mandatory in practical applications. This is because such an interface is often obtained by means of some other numerical procedure, and thus defined as a piecewise polynomial curve. This means that rigorously it is possible to compute exactly the terms $(\varphi, \mathbf{v} \cdot \overrightarrow{\nu})_\Sigma$ in (4) and (28), provided the expression of φ is simple enough. However, since this may not be the case, we derive below proper bounds for the additional error introduced in (27) and (44) when $(\varphi, \mathbf{v} \cdot \overrightarrow{\nu})_\Sigma$ is replaced with $(\varphi_h, \mathbf{v} \cdot \overrightarrow{\nu}_h)_{\Sigma_h}$ in both (4) and (28), where φ_h, Σ_h and $\overrightarrow{\nu}_h$ are suitable approximations of φ, Σ and $\overrightarrow{\nu}$. Assuming that φ is continuous on Σ, we consider φ_h to be a constant function along the chord Σ_T of Σ joining the two end points of $\Sigma \cap \bar{T}$ for each $T \in \mathcal{S}_h$. This constant value is defined in the following manner. Let M_T be the mid-point of the chord Σ_T. Taking the line r_T perpendicular to Σ_T through M_T and denoting by N_T the nearest intersection of r_T with Σ, we define φ_h over S_T to be the constant function whose value is $\varphi(N_T)$. The curve Σ itself is approximated by the polygonal line Σ_h defined as the union of the chords Σ_T over the set $\mathcal{S}_h$, and the normal $\overrightarrow{\nu}$ by the piecewise constant unit vector $\overrightarrow{\nu}_h$ whose value in every $T \in \mathcal{S}_h$ is the unit vector $\overrightarrow{\nu}_T$ normal to Σ_T and oriented in such a way that $\overrightarrow{\nu} \cdot \overrightarrow{\nu}_T > 0$.
According to well-known results the additional error is equal to a constant C_Σ multiplied by the following term:

$$E_h(\varphi, \Sigma) := \sup_{\mathbf{v} \in \mathbf{W}_h\ \mathbf{v} \neq \mathbf{0}} \frac{\sum_{T \in \mathcal{S}_h} E_T(\varphi, \Sigma, \mathbf{v})}{\| \mathbf{grad}\, \mathbf{v} \|_h} \tag{45}$$

where $\mathbf{W}_h$ equals $\mathbf{V}_{h0}^+$ in the case of (4) and $\mathbf{V}_{h0}$ in the case of (28), and the local functional E_T is defined by

$$E_T(\varphi, \Sigma, \mathbf{v}) := (\varphi, \mathbf{v}_{/\Sigma\cap T} \cdot \overrightarrow{\nu})_{\Sigma\cap T} - (\varphi_h, \mathbf{v}_{/\Sigma_T} \cdot \overrightarrow{\nu}_T)_{\Sigma_T} \tag{46}$$

Now we denote by M the orthogonal projection of an arbitrary point $P \in \Sigma$ onto Σ_h, and by z and t the local coordinates associated with $T \in \mathcal{S}_h$ respectively parallel and orthogonal to $\overrightarrow{\nu}_T$. Next we define an intermediate function $\tilde{\varphi}$ defined on Σ_T by $\tilde{\varphi}(M) = \varphi(P)$. Let us denote by v_z and v_t the functions defined in T respectively by $\mathbf{v}\cdot\overrightarrow{\nu}_T$ and the algebraic value of $\mathbf{v}\times\overrightarrow{\nu}_T$. We also denote by τ the (acute) angle between $\overrightarrow{\nu}$ and $\overrightarrow{\nu}_T$ at every point $P \in \Sigma$. Then for each element $T \in \mathcal{S}_h$, the functional E_T can be conveniently split into the sum of three terms, namely, E_{Ti}, $i = 1, 2, 3$, with

$$\begin{cases} E_{T1} = \int_{\Sigma\cap T}[\varphi v_z](P) cos\ \tau ds - \int_{\Sigma_T}[\tilde{\varphi} v_z](M) ds_T; \\ E_{T2} = \int_{\Sigma_T}[\tilde{\varphi} v_z](M) ds_T - \int_{\Sigma_T} \varphi_h v_z(M) ds_T; \\ E_{T3} = -\int_{\Sigma\cap T}[\varphi v_t](P) sin\ \tau ds. \end{cases} \tag{47}$$

Since $ds_T = cos\ \tau\ ds$ and by a Taylor expansion $v_z(M) = v_z(P) - \mathbf{grad}\ v_z \cdot \overrightarrow{MP}$, E_{T1} is readily seen to be equal to $\int_{\Sigma\cap T} \varphi \mathbf{grad}\ v_z \cdot \overrightarrow{MP} cos\ \tau\ ds$. Then since necessarily $|\overrightarrow{MP}| \leq C_g h^2$, standard manipulations using the reference element $\hat{T}$ easily yields

$$E_{T1} \leq C_{S1} h \parallel \mathbf{grad}\ \mathbf{v} \parallel_T \left| \int_{\Sigma\cap T} \varphi ds \right|. \tag{48}$$

Taking into account that the length of $\Sigma \cap T$ is to the most an $O(h)$ from (48) we derive,

$$\sum_{T\in\mathcal{S}_h} E_{T1} \leq C_{E1} h^{3/2} \parallel \mathbf{grad}\ \mathbf{v} \parallel_{\Omega_h} \parallel \varphi \parallel_\Sigma . \tag{49}$$

An estimate for E_{T3} can be easily obtained by noting that $\parallel sin\ \tau \parallel_{0,\infty,\Sigma\cap T} \leq C_\tau h$. More concretely we derive,

$$E_{T3} \leq C_{S3} h \parallel \mathbf{v} \parallel_{\Sigma\cap T} \parallel \varphi \parallel_{\Sigma\cap T} . \tag{50}$$

Summing up over $\mathcal{S}_h$ and using the Trace Theorem on Σ this immediately leads to

$$\sum_{T\in\mathcal{S}_h} E_{T3} \leq C_{E3} h \parallel \mathbf{grad}\ \mathbf{v} \parallel_{\Omega_h} \parallel \varphi \parallel_\Sigma . \tag{51}$$

Finally in order to bound the term E_{T2} we resort to the Bramble-Hilbert Lemma [9], on the segments Σ_T, assuming that $\varphi \in H^1(\Sigma)$. This assumption implies in particular that φ is uniformly continuous on Σ from the Sobolev Embedding Theorem [1]. Let then $l_T = |\Sigma_T|$ and a_T be the bilinear form given by

$$a_T(v_z, \varphi) := \int_{\Sigma_T} [\tilde{\varphi} - \varphi_h] v_z(M) ds_T.$$

Denoting by $\hat{T}$ the usual reference rectangular triangle we associate with a_T the bilinear form $\hat{a} : L^2(\hat{\Sigma}_T) \times H^1(\hat{\Sigma}_T)$, given by

$$\hat{a}(\hat{v}, \hat{\varphi}) := \int_{\hat{\Sigma}_T} [\hat{\varphi} - \hat{\varphi}(\hat{M})] v d\hat{s},$$

where $\hat{\Sigma}_T$, $\hat{\varphi}$, $\hat{M}$ and $\hat{v}$ are the images or transforms under the affine mapping taking T onto $\hat{T}$, of Σ_T, φ, M_T and v_z, respectively. Then setting $\hat{l} = |\hat{\Sigma}_T|$ we have:

$$a_T(v_z, \varphi) = \frac{l_T}{\hat{l}} \hat{a}(\hat{\varphi}, \hat{v}). \tag{52}$$

By the Trace Theorem in $\hat{T}$, and the Sobolev Embedding Theorem [1] from $H^1(\hat{\Sigma}_T)$ into $C^0(\hat{\Sigma}_T)$ equipped with the maximum norm, we clearly have,

$$\hat{a}(\hat{\varphi}, \hat{v}) \leq \hat{C}_a \parallel \hat{v} \parallel_{\hat{\Sigma}_T} \parallel \hat{\varphi} \parallel_{1,\hat{\Sigma}_T} . \tag{53}$$

Since $\hat{a}(\hat{\varphi}, \hat{v}) = 0$ whenever $\hat{\varphi}$ is constant on $\hat{\Sigma}_T$, we obtain

$$\hat{a}(\hat{\varphi}, \hat{v}) \leq \hat{C}_s \parallel \hat{v} \parallel_{\hat{\Sigma}_T} \parallel d\hat{\varphi}/d\hat{s} \parallel_{\hat{\Sigma}_T} . \tag{54}$$

Moving back to T from (52) and (54), and noticing that the ratio $l_T/\hat{l}$ is necessarily an $O(h)$ term, using standard results we obtain,

$$E_{T2} = a_T(v_z, \varphi) \leq C_{T2} h \parallel \mathbf{v} \parallel_{\Sigma_T} \parallel d\tilde{\varphi}/ds_T \parallel_{\Sigma_T} . \tag{55}$$

Summing up again over $\mathcal{S}_h$, we derive

$$\sum_{T \in \mathcal{S}_h} E_{T2} \leq C_{S2} h \parallel \mathbf{v} \parallel_{\Sigma_h} \parallel d\tilde{\varphi}/ds_h \parallel_{\Sigma_h} . \tag{56}$$

where s_h represents the abcissa along Σ_h.
Our error estimate will be complete if we prove that there exist two constants C_{Q2} and C_{R2} such that $\| \mathbf{v} \|_{\Sigma_h} \leq C_{Q2} \| \mathbf{grad}\ \mathbf{v} \|_{\Omega_h}$ and $\| d\tilde{\varphi}/ds_h \|_{\Sigma_h} \leq C_{R2} \| d\varphi/ds \|_{\Sigma}$. These bounds are a consequence of the fact that in every $T \in \mathcal{S}_h$ and $\forall M \in \Sigma_T$

$$|\mathbf{v}_{/\Sigma_T}(M)| \leq |\mathbf{v}_{/\Sigma\cap T}(P)| + C_{D2}h^2 \| \mathbf{grad}\ \mathbf{v} \|_{0,\infty,T} \leq |\mathbf{v}_{/\Sigma\cap T}(P)| + C_{E2}h \| \mathbf{grad}\ \mathbf{v} \|_T, \quad (57)$$

together with the relation

$$\int_{\Sigma_T} \left| \frac{d\tilde{\varphi}}{ds_T}(M) \right|^2 ds_T = \int_{\Sigma\cap T} [cos\ \tau]^{-1} \left| \frac{d\varphi}{ds}(P) \right|^2 ds. \quad (58)$$

After straightforward manipulations, using the inverse inequality in T and the fact that $[cos\ \tau]^{-1} \leq 1 + C_{co}h^2$, we obtain from (57),

$$\int_{\Sigma_T} |\mathbf{v}_{/\Sigma_T}(M)|^2 ds_T \leq C_{F2} \left[\int_{\Sigma\cap T} |\mathbf{v}_{/\Sigma\cap T}(P)|^2 ds + h^3 \| \mathbf{grad}\ \mathbf{v} \|_T^2 \right]. \quad (59)$$

Then summing up over T and using the Trace Theorem for Σ together with the Friedrichs-Poincaré inequality in Ω_h, (59) readily yields

$$\| \mathbf{v} \|_{\Sigma_h} \leq C_{G2}(1 + h^{3/2}) \| \mathbf{grad}\ \mathbf{v} \|_{\Omega_h}. \quad (60)$$

Finally from (58) we easily obtain

$$\| d\tilde{\varphi}/ds_h \|_{\Sigma_h} = C_{H2}(1 + h) \| d\varphi/ds \|_{\Sigma}. \quad (61)$$

Hence combining (56), (60) and (61) we derive

$$\sum_{T\in\mathcal{S}_h} E_{T2} \leq C_{E2}h \| \mathbf{grad}\ \mathbf{v} \|_{\Omega_h} \| d\varphi/ds \|_{\Sigma}. \quad (62)$$

Once established the estimates (49), (62), (51), we have actually proved the following result.

Theorem 4.1 *Assuming that $\varphi \in H^1(\Sigma)$ and provided h is small enough, the approximation of $(\varphi, \mathbf{v}\cdot\overrightarrow{\nu})_\Sigma$ by $(\varphi_h, \mathbf{v}\cdot\overrightarrow{\nu}_h)_{\Sigma_h}$ defined in this Section introduces an additional error term of the form $C_\Sigma h \| \varphi \|_{1,\Sigma}$.* ■

5 Final remarks

As a conclusion we would like first of all to make an additional comment on Assumption*. This seems to be just a technicality used here in order to prove the uniform stability of our method, rather than a necessary condition. Indeed its extensive application in the solution of wide spectra of problems defined on various types of domains and interfaces with increasing degrees of mesh refinement (see e.g. [3,8]) clearly indicate that there is no real restriction to first order convergence in the $H^1 \times L^2$ norm.
It is also worthwhile pointing out two interesting by-products of the analysis carried out in this work. The first one is the result stating that there is no harm, neither for the added-bubble nor for the added-penalty term, to the $P1$ velocity approach combined with a continuous piecewise linear pressure, if these stabilizing techniques are not employed within a band of elements with area equal to an $O(h)$. The second by-product is the fact that it is possible to work with a discontinuous linear pressure along given (polygonal) lines of $O(1)$-length immersed in the domain, provided they are the union of element edges.

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