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Finite element methods for the Stokes system based on a Zienkiewicz type N-simplex

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Dedicated to the memory of Leopoldo P. Franca and Olgierd C. Zienkiewicz

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ABSTRACT

Hermite interpolation is increasingly showing to be a powerful numerical solution tool, as applied to different kinds of second order boundary value problems. In this work we present two Hermite finite element methods to solve viscous incompressible flows problems, in both two- and three-dimension space. In the two-dimensional case we use the Zienkiewicz triangle to represent the velocity field, and in the three-dimensional case an extension of this element to tetrahedra, still called a Zienkiewicz element. Taking as a model the Stokes system, the pressure is approximated with continuous functions, either piecewise linear or piecewise quadratic, according to the version of the Zienkiewicz element in use, that is, with either incomplete or complete cubics. The methods employ both the standard Galerkin or the Petrov–Galerkin formulation first proposed in Hughes et al. (1986) [18], based on the addition of a balance of force term. A priori error analyses point to optimal convergence rates for the PG approach, and for the Galerkin formulation too, at least in some particular cases. From the point of view of both accuracy and the global number of degrees of freedom, the new methods are shown to have a favorable cost-benefit ratio, as compared to velocity Lagrange finite elements of the same order, especially if the Galerkin approach is employed.

1. Introduction

In recent years Hermite interpolation has been found to be a favorable approach to solve several kinds of field problems modeled by second order boundary value problems in different respects. A good illustration of this fact is provided by the isogeometric finite element method introduced by Hughes about ten years ago (see e.g. [9]). In this case the main motivation is the best use of data supplied by CAD in a subsequent finite element analysis using high continuity requirements. Direct representation of derivatives through Hermite interpolation is also a desired property in the simulation of phenomena, where quantities expressed in terms of derivatives of a primal variable play an important role. This is the case for instance of Hermite methods ensuring the continuity of fluxes in a porous medium flow (see e.g. [21]).

This work also addresses Hermite approaches to solve the Stokes system in a bounded domain of $\Re^N, N = 2, 3$ by the finite element method. However in contrast to isogeometric elements or Hermite analogs of mixed finite elements like the one in [22], the main goal here is not to enforce C^k continuity of the solution with $k \geqslant 1$. The main interest of the present methods relies on the particular Hermite interpolation of the velocity being used, which gives rise to rather high order methods with a

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reduced global number of degrees of freedom, as compared to standard Lagrange finite elements having equivalent approximation properties. A further aim of ours was to work with particularly simple data structures. This is actually possible for three out of the four methods to be studied, since in these cases only vertex degrees of freedom must be handled at matrix level.

More specifically we represent the velocity by means of the Zienkiewicz triangle in two-dimensional space, and of an extension of it to tetrahedra, still called a Zienkiewicz element, in the three-dimensional case. Two versions of the method are proposed in each case. In the first version the velocity is represented by incomplete cubics in each element, while a standard Lagrange interpolation of the pressure is employed with continuous piecewise linear functions. This combination is denoted by Z_2/P_1. In the second version the velocity representation is performed with complete cubics, and the pressure is approximated by means of standard continuous piecewise quadratics. This combination is denoted by Z_3/P_2.

Both approaches can be employed in connection with either a classical Petrov–Galerkin formulation, or the standard Galerkin formulation of the Stokes system. In the former case the methods are convergent with optimal orders in the natural norm of $\mathbf{H}^1(\Omega)\times L^2(\Omega)$, namely, order two for Z_2/P_1 and three for Z_3/P_2. Except for some issues related to Hermite interpolations, this can be certified in accordance with well-known results (cf. [18,10]), in the case of homogeneous Dirichlet velocity boundary conditions.

As it is well-known inhomogeneous boundary conditions are very important in practical applications related to viscous flow. However, probably because their approximation is regarded as an issue involving just a few additional technicalities, it has been rather overlooked so far (cf. [13,4] and references therein). Indeed, introducing an arbitrary field $\mathbf{w}$ satisfying the inhomogeneous boundary condition in the continuous problem, and its interpolate in the approximate problem, provided the bilinear form is uniformly bounded and stable with respect to the norm of $\mathbf{H}^1(\Omega)\times L^2(\Omega)$ in the approximation spaces, the analysis becomes only a matter of right hand side variational crime (cf. [24]). In short, the additional term in the error estimate is just the interpolation error of $\mathbf{w}$ (cf. [11]). This means that the regularity of this arbitrary field comes into play, although it may not be ensured by a non smooth boundary. That is why it is wiser to derive error estimates involving directly the interpolated boundary data, by properly identifying the scales involving both the mesh and the stabilizing parameter inherent to inhomogeneous boundary conditions. Here we achieve this by carrying out a remake of previous analyses based on a technique inspired by the classical work of Lax-Richtmyer [20]. As a by-product, on the one hand we exploit the coercivity of the bilinear form over the discretization spaces with respect to a mesh dependent norm; on the other hand we bypass both the unboundedness of the bilinear form as the mesh parameter decreases, and limitations on the stabilizing parameter. This is because we also avoid discrete inf–sup inequalities employed by most authors. Instead we complete our analysis using a celebrated result in [19].

In addition to this we supply both theoretical and numerical arguments, according to which the Z_2/P_1 method is optimally convergent in Galerkin formulation, at least in the two-dimensional case. Our numerical results also indicate that the Galerkin formulation performs better than the Petrov–Galerkin approach for this element, and also as compared to classical stable Lagrange methods of the same order, in Galerkin formulation as well.

The Stokes equations governing the slow flow of a viscous incompressible fluid with viscosity μ in a bounded domain Ω, together with pertaining notations are as follows. Γ being the boundary of Ω with unit outer normal vector $\mathbf{n}$, and given a force field $\mathbf{f}\in\mathbf{L}^2(\Omega)$, together with $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma)$ satisfying $\oint_\Gamma \mathbf{g}\cdot\mathbf{n}=0$, we wish to find a velocity $\mathbf{u}\in\mathbf{H}^1(\Omega)$ and a pressure $p\in L^2_0(\Omega)$ (cf. [14]) such that:

$$\begin{cases} -\mu\Delta\mathbf{u}+\mathbf{grad}\,p=\mathbf{f} & \text{in } \Omega,\\ div\,\mathbf{u}=\mathbf{0} & \text{in } \Omega,\\ \mathbf{u}=\mathbf{g} & \text{on } \Gamma. \end{cases} \tag{1}$$

Denoting by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ in scalar, vector or tensor version, with associated norm $\|\cdot\|$, and by $(\cdot,\cdot)_D$ the standard inner product of $L^2(D)$ with associated norm $\|\cdot\|_D$, for any $D\subsetneq\Omega$, we may rewrite problem (1) in the following equivalent variational formulation:

$$\begin{cases} \text{Find} \quad \mathbf{u}\in\mathbf{H}^1(\Omega) \text{ and } p\in L^2_0(\Omega) & \text{such that,}\\ \mu(\mathbf{grad}\,\mathbf{u},\mathbf{grad}\,\mathbf{v})-(p,div\,\mathbf{v})=(\mathbf{f},\mathbf{v}) & \forall\mathbf{v}\in\mathbf{H}^1_0(\Omega),\\ (div\,\mathbf{u},q)=0 & \forall q\in L^2_0(\Omega),\\ \mathbf{u}=\mathbf{g} & \text{on } \Gamma. \end{cases} \tag{2}$$

Throughout this work we use the following notations: $\|\cdot\|_{r,D}$ is the standard norm of Sobolev space $H^r(D)$ for $r\in\Re$ and $|\cdot|_{m,D}$ represents the standard semi-norm of Sobolev space $H^m(D)$, for $m\in\mathbb{N}$, D being a subset of Ω. We drop the subscript D whenever D is Ω itself.

An outline of the paper is as follows. Section 2 is devoted to the description of the Hermite finite element methods in the N- dimensional case for $N=2$ and $N=3$. In Section 3 we set up the Z_2/P_1 and the Z_3/P_2 approximation methods for solving system (1), in both Petrov–Galerkin and standard Galerkin formulation. An a priori error analysis of the former type of formulation is given in Section 4, applying to the case of inhomogeneous velocity Dirichlet boundary conditions. Next, convergence results are given in Section 5 for the Galerkin formulation, restricted to the case of the two-dimensional Z_2/P_1 method on criss-cross meshes. Section 6 is aimed at illustrating the performance of the new methods from the numerical point of view, by means of two-dimensional test-problems. We conclude in Section 7 with some remarks.

2. Finite element description

To begin with we specify the Hermite finite elements we use to represent the velocity field locally, that is in every N-simplex T of a mesh, $N = 2, 3$.

Let S_i be the vertices of $T, i = 1, \ldots, N+1$, and G its barycenter. We denote by λ_i the barycentric coordinate of T associated with S_i and set

$$h_{ij} = \text{length}\,[S_iS_j].$$

The reader is referred to Fig. 1 for an illustration of the degree of freedom structure of each kind of velocity interpolation in use, as described hereafter.

In the case $N = 2$ the elements are nothing but the well-known Zienkiewicz triangle in its two versions referred to here as Z_2 and Z_3, that is, with either incomplete or complete cubics, as defined in [27]. For better guidance we recall below that the nine degrees of freedom of Z_2 are the function values and the first order derivatives along the edges of T at its three vertices. For a convenient description of this Hermite finite element, the derivative along a given edge at a vertex belonging to it is always taken in the direction leading from this vertex to the other end of the edge under consideration. Denoting the bubble function of T by

$$\varphi = \lambda_1\lambda_2\lambda_3$$

and $P_m(T)$ being the space of polynomials of degree less than or equal to m defined in T, the subspace of $P_3(T)$ associated with Z_2 is the one spanned by the set of nine linearly independent functions $[\{\zeta_i\}_{i=1}^3 \cup \{\zeta_{ij}\}_{i\neq j=1}^3]$, where

$$\zeta_i = \lambda_i^3 - \varphi \quad \text{and} \quad \zeta_{ij} = \lambda_i^2\lambda_j + \varphi/2.$$

The nine canonical basis functions corresponding to the above specified degrees of freedom are:

- For the function value at $S_i : \varphi_i := 3\lambda_i^2 - 2\zeta_i, i \in \{1,2,3\}$;
- For the derivative at S_i in the direction of $\overrightarrow{S_iS_j} : \varphi_{ij} := h_{ij}\zeta_{ij},\ i,j \in \{1,2,3\}, i \neq j$.

In the case of Z_3 the above set of degrees of freedom is augmented with the function value at G. The set of ten basis functions associated with Z_3 are:

- For the function value at $S_i : \varphi_i := 3\lambda_i^2 - 2\zeta_i - 9\varphi, i \in \{1,2,3\}$;
- For the derivative at S_i in the direction of $\overrightarrow{S_iS_j} : \varphi_{ij} := h_{ij}(\zeta_{ij} - 3\varphi/2),\ i,j \in \{1,2,3\}, i \neq j$;
- For the function value at $G : \varphi_{123} := 27\varphi$.

The extension to tetrahedra of the Zienkiewicz triangle we consider in this work is described below:

Denoting by φ_{ijk} the bubble function $\lambda_i\lambda_j\lambda_k$ of face F_l of T, where the integers $i, j, k, l \in \{1,2,3,4\}$ are assumed to be distinct, the analog of Z_2 still denoted this way is spanned by the set of sixteen linearly independent functions $[\{\zeta_i\}_{i=1}^4 \cup \{\zeta_{ij}\}_{i\neq j=1}^4]$, where

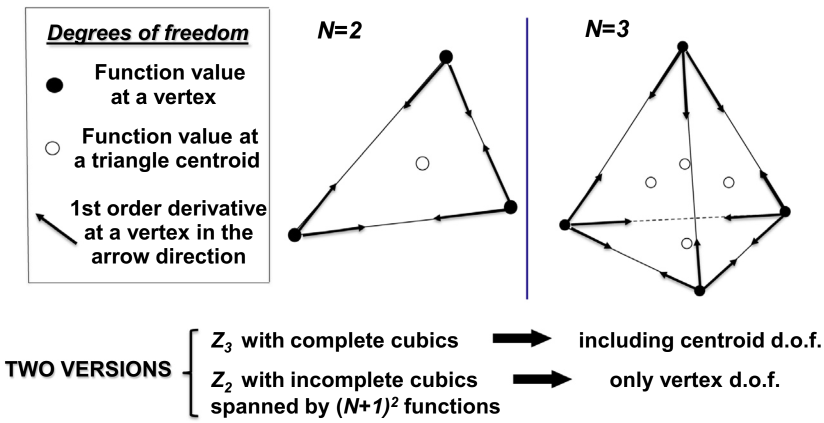

Fig. 1. Degrees of freedom of each velocity component in spaces Z_2 and Z_3.

$$\zeta_i = \lambda_i^3 - \varphi_{ijk} - \varphi_{ijl} - \varphi_{ikl} \quad \text{and} \quad \zeta_{ij} = \lambda_i^2 \lambda_j + (\varphi_{ijk} + \varphi_{ijl})/2.$$

The set of sixteen degrees of freedom defining Z_2 in connection with the above basis are the function values at S_i and the first order derivatives at S_i along the three edges converging to this point, for $i = 1, 2, 3, 4$. Using the same convention concerning the sense of these derivatives as in the two-dimensional case, the set of canonical basis functions associated with such degrees of freedom are given by:

- For the function value at $S_i : \varphi_i := 3\lambda_i^2 - 2\zeta_i, i \in \{1, 2, 3, 4\}$;
- For the derivative at S_i in the direction of $\overrightarrow{S_iS_j} : \varphi_{ij} := h_{ij}\zeta_{ij}, i,j \in \{1, 2, 3, 4\}, i \neq j$.

Using the same notation for the analog of Z_3, this element is based on the space $P_3(T)$. The dimension of this space being twenty, the set of degrees of freedom defining Z_3 in connection with it are the function values at S_i and at the centroid G_i of the face F_i opposite to S_i, together with the first order derivatives at S_i along the three edges converging to this point, for $i = 1, 2, 3, 4$. The set of canonical basis functions associated with such degrees of freedom are given by:

- For the function value at $S_i : \varphi_i := 3\lambda_i^2 - 2\zeta_i - 9(\varphi_{ijk} + \varphi_{ijl} + \varphi_{ikl})$;
- For the derivative at S_i in the direction of $\overrightarrow{S_iS_j} : \varphi_{ij} := h_{ij}[\zeta_{ij} - 3(\varphi_{ijk} + \varphi_{ijl})/2]$;
- For the function value at $G_i : \varphi_i^G := 27\varphi_{jkl}$.

The approximation properties of the above two- and three- dimensional elements were studied in [7,25] respectively. Let us briefly recall them.

First of all it is an easy matter to verify that the subspaces of $P_3(T)$ for elements Z_2 contain the space $P_2(T)$. Therefore if u is a function in $H^{l+1}(T)$, we can assert that the Z_l-interpolate of u in T, denoted by $\pi_T^l(u)$, satisfies for suitable constants C_m^l independent of T and u,

$$\|u - \pi_T^l(u)\|_{m,T} \leqslant C_m^l h^{l+1-m} |u|_{l+1,T} \; m = 0, 1, \ldots, l + 1.$$

Although the three-dimensional elements have been previously quoted in the literature (see e.g. [7]) one cannot say that they are well-known. Therefore we endeavor to check here that the corresponding sets of basis functions satisfy the required conditions to be canonical. In this aim we first note that the function values together with the first order derivatives of all the face bubble functions vanish at the vertices of T, as does the bubble function φ and **grad** φ if T is a triangle. This means that we can disregard terms of the above basis set involving these bubble functions.

Let us first consider the case of Z_2:

As far as φ_i is concerned we note that $\varphi_i(S_j) = \delta_{ij}$. Moreover since $\lambda_i(1 - \lambda_i)$ is a factor of the gradient of φ_i (without bubbles), it vanishes at all the vertices of T. It follows that φ_i is indeed the canonical basis function associated with the function value at vertex S_i. The function φ_{ij}, in turn, vanishes at every vertex of T. Moreover since λ_i is a factor of **grad** φ_{ij} (without bubbles), its value at all the vertices of F_i vanishes. Finally the derivative of φ_{ij} at S_i in the direction of the edges S_iS_k or S_iS_l is simply h_{ij} times the derivative of λ_j in these directions. Since λ_j vanishes identically along both edges, these derivatives are equal to zero. Finally we note that the derivative of φ_{ij} at S_i in the direction $\overrightarrow{S_iS_j}$ is given by h_{ij} times the derivative of λ_j in this direction, that is $1/h_{ij}$ and we are done.

From this point the check procedure of the basis functions for element Z_3 becomes trivial. Indeed the functions added to the basis functions of Z_2 are just aimed at making them vanish at the four face centroids, and moreover the basis functions φ_i^G have the same property, but at G_i where its value equals one.

To conclude this Section it is worth pointing out that the above degrees of freedom related to derivatives at a given vertex are not suitable for computations in the framework of a finite element solution. This is because in general it is not possible to let them coincide for all the elements intersecting at this vertex. However we may combine linearly the corresponding basis functions in each N-simplex in order to construct canonical basis functions associated with the first order partial derivatives in the two or three fixed directions of the cartesian axes, that is, the ones of unit vectors $\mathbf{e}^m$, $m = 1, \ldots, N$ forming an orthonormal basis of $\Re^N$. In doing so the assembly of the element matrices will lead to the correct global matrix referred to the derivatives in these directions only, at all the vertices of the mesh. More concretely we supply below the expressions of the basis functions φ_i^m pertaining to Z_2, such that $[\mathbf{grad}\, \varphi_i^m \cdot \mathbf{e}^n](S_j) = \delta_{ij}\delta_{mn}$ and $\varphi_i^m(S_j) = 0$ for $i,j \in \{1, \ldots, N+1\}$ and $m, n \in \{1, \ldots, N\}$: For distinct i,j and denoting by $(x_i^1, \ldots, x_i^N)$ the cartesian coordinates of S_i, $i = 1, \ldots, N+1$,

$$\varphi_i^m = \sum_{j=1,\, j \neq i}^{N+1} \varphi_{ij}(x_j^m - x_i^m). \tag{3}$$

Naturally enough the vertex value basis functions φ_i do not require any modification, owing to the fact that their gradients vanish at all the vertices by construction. Moreover since the same property holds for the triangle bubble or face bubble functions, the expression (3) also applies to the basis functions φ_i^m in the case of the space of complete cubics.

3. Solution with Zienkiewicz-type elements

Henceforth we assume that Ω is a polygon if $N = 2$ and a polyhedron if $N = 3$.

Before proceeding a word of clarification is in order: Using element by element inverse inequalities (cf. [7]) rather than the global ones employed hereafter, the convergence results to be given in this paper extend to the case of shape-regular families of meshes. Of course in this case, instead of the single parameter δ introduced below, one has to use element-dependent ones like in [18]. However this brings about substantially more complicated notations and expressions in the approximate problems. That is why, similarly to [12], we assume here that we are using a quasi-uniform family of partitions $\mathcal{T}_h$ of Ω into triangles or tetrahedra, satisfying the compatibility conditions for finite element meshes specified in classical references on the subject such as [7,11].

Let h denote the maximum edge length of the elements in $\mathcal{T}_h$. In all the sequel the letter C combined or not with other symbols represents constants independent of h. Also throughout this work $\mathbf{g}_h$ stands for piecewise cubic Hermite interpolates of $\mathbf{g}$ on Γ, assuming henceforth that $\mathbf{g}_{|\Gamma_i}$ belongs to $\mathbf{H}^{5/2}(\Gamma_i)$ for $i = 1, 2, \ldots, m$, where the Γ_i's are the straight edges or the plane faces of Γ. More specifically if $N = 2$ we mean the classical cubic interpolate at the vertices of $\mathcal{T}_h$ belonging to Γ, continuously differentiable along every straight portion of Γ provided $\mathbf{g}_{|\Gamma_i} \in \mathbf{H}^{5/2}(\Gamma_i)$. If $N = 3$ we define $\mathbf{g}_h$ to be the cubic Hermite interpolate of $\mathbf{g}$ on every face contained in Γ_i of a tetrahedron of $\mathcal{T}_h$, using the degrees of freedom of the Zienkiewicz triangle, either with complete or incomplete cubics, according to the method being studied. Assuming that $\mathbf{f} \in \mathbf{H}^{l+1}(T)$ in every $T \in \mathcal{T}_h$, for l equal to 1 or 2, we will also work with approximations $\mathbf{f}_h^l$ of $\mathbf{f}$ in every element of $\mathcal{T}_h$, satisfying $\|\mathbf{f} - \mathbf{f}_h^l\|_T \leqslant C_l h^{l+1} |\mathbf{f}|_{l+1,T}\ \forall T \in \mathcal{T}_h$.

We shall consider problems to approximate (1) of the same kind as those proposed by Hughes–Franca-Balestra [18] and Douglas–Wang [10]. However in contrast to those works, our convergence analysis will be based on the Lax Equivalence Theorem [20] combined with Ladyzhenskaia's condition for the divergence operator [19], for the reasons mentioned in the third paragraph of Section 1.

The solution methods to be studied use the pressure space Q_h^l, for $l = 1, 2$ respectively, defined as follows:

$$Q_h^l := \{q \,/\, q \in C^0(\bar{\Omega}) \cap L_0^2(\Omega),\ q_{/T} \in P_l(T)\ \forall T \in \mathcal{T}_h\}$$

We associate with Q_h^l spaces $\mathbf{V}_h^l := [V_h^l]^N$ for $l = 1, 2$ to represent the velocity, both being constructed upon the plate Zienkiewicz element Z_{l+1} for $N = 2$ [27] or with its three-dimensional version defined in Section 2. This means that V_h^l is a space of continuous functions of degree less than or equal to three in each element of $\mathcal{T}_h$, whose gradient is also continuous at their vertices. While on the one hand V_h^2 consists of piecewise complete cubics in every N-simplex of $\mathcal{T}_h$, on the other hand in every element T a function of V_h^1 belongs to the space spanned by $(N+1)^2$ cubic functions in T, such that it does not contain either multiples of the triangle bubble function if $N = 2$ or linear combinations of the four face bubble functions if $N = 3$.

Let us denote by V_{h0}^l the space $V_h^l \cap H_0^1(\Omega)$, and introduce the following broken $L^2(\Omega)$-inner product denoted by $(\cdot,\cdot)_h$, with associated norm $\|\cdot\|_h$, for functions u and v defined only in the interior of the elements of $\mathcal{T}_h$:

$$(u, v)_h = \sum_{T \in \mathcal{T}_h} (u, v)_T; \quad \|v\|_h = \sqrt{(v, v)_h}.$$

Now given a numerical parameter $\delta > 0$ to be specified later on, we consider the following problems to approximate (2), where l equals 1 or 2:

$$\begin{cases} \text{Find} \quad \mathbf{u}_h^l \in \mathbf{V}_h^l \text{ and } p_h^l \in Q_h^l \text{ such that } \forall \mathbf{v} \in \mathbf{V}_{h0}^l \text{ and } \forall q \in Q_h^l \\ \delta(\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\, p_h^l, \mu\Delta\mathbf{v} - \mathbf{grad}\, q)_h + \mu(\mathbf{grad}\,\mathbf{u}_h^l, \mathbf{grad}\,\mathbf{v}) - (p_h^l, div\,\mathbf{v}) + (div\,\mathbf{u}_h^l, q) \\ = -\delta(\mathbf{f}_h^l, \mu\Delta\mathbf{v} - \mathbf{grad}\, q)_h + (\mathbf{f}_h^l, \mathbf{v}) \\ \mathbf{u}_h^l = \mathbf{g}_h \text{ on } \Gamma. \end{cases} \tag{4}$$

Remark 3.1. In problem (4) we chose the Petrov–Galerkin formulation with a modification à la Douglas-Wang [10] rather than the original one proposed in [18]. The latter is derived by changing the sign of the viscous test term $\mu\Delta\,\mathbf{v}$, and in this case we obtain a symmetric problem. However as pointed out in [13], such a choice induces restrictions on δ. As seen hereafter this is completed avoided, in case formulation (4) is employed (see also [10] for the case of homogeneous velocity boundary conditions). □

Proposition 3.1. *Problem (4) has a unique solution.*

Proof. First of all we prove the solution uniqueness. Since we are dealing with a linear problem it suffices to establish that if we take zero data in (4), i.e. $\mathbf{g}_h = \mathbf{0}$ and $\mathbf{f}_h^l = \mathbf{0}$, the solution vanishes identically. In this case we may take $\mathbf{v} = \mathbf{u}_h^l$ and $q = p_h^l$, thereby obtaining:

$$\mu\|\mathbf{grad}\,\mathbf{u}_h^l\|^2 + \delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\,p_h^l\|_h^2 = 0.$$

Hence $\mathbf{grad}\,\mathbf{u}_h^l = \mathcal{O}$, which implies that $\mathbf{u}_h^l = \mathbf{0}$ in every element of $\mathcal{T}_h$. It follows that $\mathbf{grad}\,p_h^l = \mathbf{0}$ in every $T \in \mathcal{T}_h$ too, which implies in turn that $p_h^l = 0$.

Now the existence of a solution follows from the well-known fact that uniqueness of the solution to a linear system of M equations having exactly M unknowns like (4), is equivalent to existence of a solution for arbitrary data. This completes the proof. □

To conclude this Section we note that the standard Galerkin formulation of the problem solved with the elements Z_2/P_1 or Z_3/P_2 is deduced from (4) by taking $\delta = 0$. It is well-known that in this case the existence and uniqueness of a solution relies on the validity of an inf–sup condition relating the velocity and pressure spaces (see e.g. [6]). In Section 5 we shall address this issue more thoroughly.

4. A study of the Petrov–Galerkin approach

In this Section we study problem (4) using a technique of analysis aimed at deriving a priori error estimates in terms of $\|\mathbf{g} - \mathbf{g}_h\|_\Gamma$, i.e. the interpolation error of inhomogeneous velocity boundary conditions. We also incorporate to our approach a way to bypass Clément or Scott–Zhang type operators used in most studies prior to this work, since to the best of our knowledge none is available for Hermite elements.

4.1. Stability results

To begin with we derive stability results for the problem under study. Without loss of essential results, like many authors do, we assume henceforth that we are working with dimensionless lengths. In this way we may consider that $h < 1$. For the purpose of our stability analysis, it is convenient to consider a system of the same form, though more general than (4), namely:

$$\begin{cases} \text{Find} \quad \mathbf{u}_h^l \in \mathbf{V}_h^l \text{ and } p_h^l \in Q_h^l \text{ such that } \forall \mathbf{v} \in \mathbf{V}_{h0}^l \text{ and } \forall q \in Q_h^l \\ \delta(\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\,p_h^l, \mu\Delta\mathbf{v} - \mathbf{grad}\,q)_h + \mu(\mathbf{grad}\,\mathbf{u}_h^l, \mathbf{grad}\,\mathbf{v}) - (p_h^l, div\,\mathbf{v}) + (div\,\mathbf{u}_h^l, q) \\ = E(\mathbf{grad}\,\mathbf{v}) + F(\mathbf{v}) + L(\mu\Delta\mathbf{v} - \mathbf{grad}\,q) + P(\mathbf{grad}\,q) + R(\gamma[q]) \\ \mathbf{u}_h^l = \mathbf{g}_h \text{ on } \Gamma. \end{cases} \tag{5}$$

where $\gamma[v]$ is the trace on Γ of a function $v \in H^1(\Omega)$, and E, F, R, L, P are linear functionals satisfying:

$$\begin{aligned} & E(\mathbf{grad}\,\mathbf{v}) \leqslant |E|\|\mathbf{grad}\,\mathbf{v}\| \;\forall \mathbf{v} \in \mathbf{V}_h^l \\ & F(\mathbf{v}) \leqslant |F|\|\mathbf{grad}\,\mathbf{v}\| \;\forall \mathbf{v} \in \mathbf{V}_h^l \\ & L(\mu\Delta\mathbf{v} - \mathbf{grad}\,q) \leqslant |L|\|\mu\Delta\mathbf{v} - \mathbf{grad}\,q\|_h \;\forall \mathbf{v} \in \mathbf{V}_h^l \text{ and } \forall q \in Q_h^l \\ & P(\mathbf{grad}\,q) \leqslant |P|\|\mathbf{grad}\,q\| \;\forall q \in Q_h^l \\ & R(\gamma(q)) \leqslant |R|\|\mathbf{grad}\,q\| \;\forall q \in Q_h^l. \end{aligned} \tag{6}$$

which can be seen as a definition of $|E|, |F|, |R|, |L|$ and $|P|$. We note that some of these quantities may depend on h.

Proposition 4.1. *Let $\mathbf{w}_h$ be the unique field of $\mathbf{V}_h^l$ that minimizes the functional $\|\mathbf{grad}\,\mathbf{v}\|^2$ under the constraint $\mathbf{v} = \mathbf{g}_h$ on Γ. Then the following stability result holds for problem (5):*

$$\delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\,p_h^l\|_h^2 + \mu\|\mathbf{grad}\,\mathbf{u}_h^l\|^2 \leqslant \widetilde{C}^l\left[\delta\|\Delta\mathbf{w}_h\|_h^2 + \left(\frac{1}{\delta} + \frac{1}{h^2}\right)\|\mathbf{grad}\,\mathbf{w}_h\|^2 + (|E| + |F|)^2 + \frac{|L|^2}{\delta} + (|P| + |R|)^2\left(\frac{1}{\delta} + \frac{1}{h^2}\right)\right]. \tag{7}$$

Proof. First we set in (5) $\mathbf{v} = \mathbf{u}_h^l - \mathbf{w}_h \in \mathbf{V}_{h0}^l$ and $q = p_h^l$. In doing so we come up with:

$$\begin{aligned} \delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\,p_h^l\|_h^2 + \mu\|\mathbf{grad}\,\mathbf{u}_h^l\|^2 = & -(p_h^l, div\,\mathbf{w}_h) + \delta(\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\,p_h^l, \mu\Delta\mathbf{w}_h)_h + \mu(\mathbf{grad}\,\mathbf{u}_h^l, \mathbf{grad}\,\mathbf{w}_h)_h \\ & + E(\mathbf{grad}[\mathbf{u}_h^l - \mathbf{w}_h]) + F(\mathbf{u}_h^l - \mathbf{w}_h) + L(\mu\Delta[\mathbf{u}_h^l - \mathbf{w}_h] - \mathbf{grad}\,p_h^l) + P(\mathbf{grad}\,p_h^l) \\ & + R(\gamma[p_h^l]). \end{aligned} \tag{8}$$

Letting C_B be a constant fulfilling the inequality (see e.g. [5]),

$$\|q\| \leqslant C_B\|\mathbf{grad}\,q\| \;\forall q \in H^1(\Omega) \cap L_0^2(\Omega), \tag{9}$$

from (8) straightforward calculations successively yield:

$$\delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\ p_h^l\|_h^2 + \mu\|\mathbf{grad}\ \mathbf{u}_h^l\|^2 \leqslant \delta\|\mu\Delta\mathbf{w}_h\|_h^2 + \mu\|\mathbf{grad}\ \mathbf{w}_h\|^2 + 2\left[C_B\|div\ \mathbf{w}_h\|\|\mathbf{grad}\ p_h^l\| + (|E|+|F|)\|\mathbf{grad}\ (\mathbf{u}_h^l - \mathbf{w}_h)\| + |L|\|\mu\Delta(\mathbf{u}_h^l - \mathbf{w}_h) - \mathbf{grad}\ p_h^l\|_h + (|P|+|R|)\|\mathbf{grad}\ p_h^l\|\right] \text{ and,}$$

$$\frac{\delta}{2}\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\ p_h^l\|_h^2 + \frac{\mu}{2}\|\mathbf{grad}\ \mathbf{u}_h^l\|^2 \leqslant \frac{3\mu}{2}\|\mathbf{grad}\ \mathbf{w}_h\|^2 + \delta\mu^2\|\Delta\mathbf{w}_h\|_h^2 + \frac{3(|E|+|F|)^2}{\mu} + 2\left[\mu|L|\|\Delta\ \mathbf{w}_h\|_h + \frac{|L|^2}{\delta} + (C_B\sqrt{N}\|\mathbf{grad}\ \mathbf{w}_h\| + |P| + |R|)(\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\ p_h^l\|_h + \|\mu\Delta\ \mathbf{u}_h^l\|_h)\right]. \quad (10)$$

Moreover from the classical inverse inequality [11], there exists C_I such that,

$$h\|\Delta\mathbf{v}\|_h \leqslant C_I\|\mathbf{grad}\ \mathbf{v}\|\ \forall \mathbf{v} \in \mathbf{V}_h^l. \quad (11)$$

Hence after simple calculations (11) together with (10) lead to,

$$\delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\ p_h^l\|_h^2 + \mu\|\mathbf{grad}\ \mathbf{u}_h^l\|^2 \leqslant 6\mu\|\mathbf{grad}\ \mathbf{w}_h\|^2 + 4\delta\mu^2\|\Delta\mathbf{w}_h\|_h^2 + \frac{12(|E|+|F|)^2}{\mu} + 8\left[\mu|L|\|\Delta\mathbf{w}_h\|_h + \frac{|L|^2}{\delta} + 2\left(C_B\sqrt{N}\|\mathbf{grad}\ \mathbf{w}_h\| + |P| + |R|\right)^2\left(\frac{1}{\delta} + \frac{\mu C_I^2}{h^2}\right)\right] \quad (12)$$

and inequality (7) readily follows from (12). □

As a consequence of Proposition 4.1 we have:

Theorem 4.1. *The unique solution of problem (4) satisfies the following stability property for a suitable constant $\widehat{C}^l$, as long as h is sufficiently small:*

$$\delta\|\mu\Delta\mathbf{u}_h^l - \mathbf{grad}\ p_h^l\|_h^2 + \mu\|\mathbf{grad}\ \mathbf{u}_h^l\|^2 \leqslant \widehat{C}^l\left[\left(\frac{\delta}{h^2} + \frac{1}{\delta} + \frac{1}{h^2}\right)\|\mathbf{g}_h\|_{1/2,\Gamma}^2 + (1+\delta)\|\mathbf{f}_h^l\|^2\right]. \quad (13)$$

Proof. Let $\tilde{\mathbf{w}}$ be the harmonic field in $\mathbf{H}^1(\Omega)$ whose trace on Γ is $\mathbf{g}_h$. We know that $\|\mathbf{grad}\ \tilde{\mathbf{w}}\| \leqslant C_H\|\mathbf{g}_h\|_{1/2,\Gamma}$ (see e.g. [16]). Moreover $(\mathbf{grad}[\tilde{\mathbf{w}} - \mathbf{w}_h], \mathbf{grad}\ \mathbf{v}) = 0\ \forall \mathbf{v} \in \mathbf{V}_{h0}^l$, from the definition of $\mathbf{w}_h$. Let also $\mathbf{w}_g$ be a field in $\mathbf{V}_h^l$ whose trace on Γ is $\mathbf{g}_h$. Setting $\mathbf{u}_h := \mathbf{w}_h - \mathbf{w}_g$ and $\tilde{\mathbf{u}} := \tilde{\mathbf{w}} - \mathbf{w}_g$ by Céa's Lemma we have $\|\mathbf{grad}(\tilde{\mathbf{w}} - \mathbf{w}_h)\| = \|\mathbf{grad}(\tilde{\mathbf{u}} - \mathbf{u}_h)\| \leqslant \inf_{\mathbf{v}\in\mathbf{V}_{h0}^l}\|\mathbf{grad}(\tilde{\mathbf{u}} - \mathbf{v})\|$. Let $\{\mathbf{u}_n\}_n$ be a sequence in $\mathbf{H}^3(\Omega) \cap \mathbf{H}_0^1(\Omega)$ that converges to $\tilde{\mathbf{u}}$ in $\mathbf{H}_0^1(\Omega)$. Setting $\varepsilon = \|\mathbf{grad}\ \tilde{\mathbf{w}}\|$ we take n such that $\|\mathbf{grad}(\mathbf{u}_n - \tilde{\mathbf{u}})\| \leqslant \varepsilon/2$, and owing to the Sobolev Embedding Theorem [1] we may choose $\mathbf{v}$ to be the interpolate of $\mathbf{u}_n$ in $\mathbf{V}_{h0}^l$. Then from standard approximation results (cf. [7]), for h small enough, we have $\|\mathbf{grad}(\mathbf{u}_n - \mathbf{v})\| \leqslant \varepsilon/2$, and thus $\|\mathbf{grad}(\mathbf{w}_h - \tilde{\mathbf{w}})\| \leqslant \|\mathbf{grad}\ \tilde{\mathbf{w}}\|$. It follows that $\|\mathbf{grad}\ \mathbf{w}_h\| \leqslant 2\|\mathbf{grad}\ \tilde{\mathbf{w}}\|$, provided h is sufficiently small, and hence in this case there exists a constant C_g such that

$$\|\mathbf{grad}\ \mathbf{w}_h\| \leqslant C_g\|\mathbf{g}_h\|_{1/2,\Gamma}; \quad h\|\Delta\mathbf{w}_h\|_h \leqslant C_g C_I\|\mathbf{g}_h\|_{1/2,\Gamma}. \quad (14)$$

Next we note that problem (4) is of the form (5) with $|E| = |P| = |R| = 0$ and $F(\mathbf{z}) = (\mathbf{f}_h^l, \mathbf{z}), L(\mathbf{z}) = \delta(\mathbf{f}_h^l, \mathbf{z})$, so that $|L| = \delta\|\mathbf{f}_h^l\|$ and $|F| = C_p\|\mathbf{f}_h^l\|, C_p$ being a constant satisfying the Friedrichs-Poincaré inequality $\|v\| \leqslant C_p\|\mathbf{grad}\ v\|\ \forall v \in H_0^1(\Omega)$.

Using (14) and taking the above values of the functional norms $|E|, |P|, |R|, |F|, |L|$ into (7) the result follows after straightforward calculations. □

4.2. A priori error estimates

In order to prove that both methods proposed in this work converge with optimal orders in Petrov–Galerkin formulation for sufficiently smooth inhomogeneous boundary conditions, we next establish the consistency of (4). In this aim we will assume that $\mathbf{u} \in \mathbf{H}^{l+2}(\Omega)$ and $p \in H^{l+1}(\Omega)$, so that we can define $\tilde{\mathbf{u}}_h^l \in \mathbf{V}_h^l$ and $\tilde{p}_h^l \in Q_h^l$ as the standard $\mathbf{V}_h^l$-interpolate of $\mathbf{u}$ and the Q_h^l-interpolate of p, respectively (cf. [7]). Then we further define the pair $(\bar{\mathbf{u}}_h^l; \bar{p}_h^l) := (\tilde{\mathbf{u}}_h^l; \tilde{p}_h^l) - (\mathbf{u}_h^l; p_h^l)$, $(\mathbf{u}_h^l; p_h^l)$ being the solution of (4).

The following preliminary result can be established for $(\bar{\mathbf{u}}_h^l; \bar{p}_h^l)$:

Proposition 4.2. *The pair $(\bar{\mathbf{u}}_h^l; \bar{p}_h^l)$ is the solution of a problem of the form (5) with the following functionals on the right hand side, where I denotes the identity tensor:* $F(\mathbf{v}) = (\mathbf{f} - \mathbf{f}_h^l, \mathbf{v}), E(\mathbf{grad}\ \mathbf{v}) = (\mu\,\mathbf{grad}\ [\tilde{\mathbf{u}}_h - \mathbf{u}] - [\tilde{p}_h - p]I, \mathbf{grad}\ \mathbf{v}),\ P(\mathbf{z}) = -(\tilde{\mathbf{u}}_h^l - \mathbf{u}, \mathbf{z})$, $L(\mathbf{z}) = \delta(\mu\Delta[\tilde{\mathbf{u}}_h^l - \mathbf{u}] - \mathbf{grad}[\tilde{p}_h^l - p] - \mathbf{f}_h^l + \mathbf{f}, \mathbf{z})$, $R(\gamma[q]) = ([\mathbf{g}_h - \mathbf{g}]\cdot\mathbf{n}, \gamma[q])$.

Proof. First we replace $(\mathbf{u}_h^l; p_h^l)$ with $(\tilde{\mathbf{u}}_h^l; \tilde{p}_h^l)$ on the left hand side of the first two equations of (4). Adding and subtracting the exact solution $(\mathbf{u}; p)$ on the resulting left hand side and making use of both (2) and (1) together with the identity $(q, div\ \mathbf{v}) \equiv (qI, \mathbf{grad}\,\mathbf{v})$ and the Divergence Theorem, we obtain:

$$\begin{cases} \forall \mathbf{v} \in \mathbf{V}_{h0}^l \text{ and } \forall q \in Q_h^l, \\ \delta(\mu\Delta\tilde{\mathbf{u}}_h^l - \mathbf{grad}\, \tilde{p}_h^l, \mu\Delta\mathbf{v} - \mathbf{grad}\, q)_h + \mu(\mathbf{grad}\, \tilde{\mathbf{u}}_h^l, \mathbf{grad}\, \mathbf{v}) - (\tilde{p}_h^l, div\, \mathbf{v}) + (div\, \tilde{\mathbf{u}}_h^l, q) \\ = (\mu\mathbf{grad}\, [\tilde{\mathbf{u}}_h^l - \mathbf{u}] - [\tilde{p}_h^l - p]I, \mathbf{grad}\, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \\ +\delta(\mu\Delta[\tilde{\mathbf{u}}_h^l - \mathbf{u}] - \mathbf{grad}[\tilde{p}_h^l - p] - \mathbf{f}, \mu\Delta\mathbf{v} - \mathbf{grad}\, q)_h - (\tilde{\mathbf{u}}_h^l - \mathbf{u}, \mathbf{grad}\, q) + ([\mathbf{g}_h - \mathbf{g}] \cdot \mathbf{n}, \gamma[q]) \end{cases} \tag{15}$$

Now taking $(\mathbf{u}_h^l; p_h^l)$ to satisfy (4), by linearity the result immediately follows. □

In view of Proposition 4.2 we can apply the stability result (7), thereby establishing the consistency of the approximate problem (4) according to:

Proposition 4.3. *Assume that $\mathbf{f} \in \mathbf{H}^{l+1}(\Omega), \mathbf{u} \in \mathbf{H}^{l+2}(\Omega), p \in H^{l+1}(\Omega)$ and $\mathbf{g}_{/\Gamma_i} \in \mathbf{H}^{l+2}(\Gamma_i)$ for $i = 1, 2, \ldots, m$, where the Γ_i's are the m disjoint straight edges for $N = 2$ or plane faces for $N = 3$, whose union is Γ. If $\mathbf{f}_h^l$ is chosen such that $\|\mathbf{f} - \mathbf{f}_h^l\| \leqslant C_l h^{l+1} |\mathbf{f}|_{l+1}$, and we take $\delta = h^2$, then there exist constants $\overline{C}^l$ such that*

$$h\|\mu\Delta\bar{\mathbf{u}}_h^l - \mathbf{grad}\, \bar{p}_h^l\|_h + \|\mathbf{grad}\, \bar{\mathbf{u}}_h^l\| \leqslant \overline{C}^l h^{l+1} \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right]. \tag{16}$$

Proof. Since $\bar{\mathbf{u}}_h^l$ vanishes on Γ the corresponding field $\mathbf{w}_h$ is zero. Therefore the first three terms in brackets are a consequence of (7) and standard estimates, as far as the functionals E, F, L, P defined in Proposition 4.2 are concerned. On the other hand using the Trace Theorem [1] and (9), well-known estimates for the interpolation error of $\mathbf{g}$ in the norm of $L^2(\Gamma_i)$, in connection with functional R also defined therein, complete the proof. □

Finally we can derive some a priori error estimates for problem (4). First of all a simple application of the triangular inequality to (16) gives,

Theorem 4.2. *If $\delta = h^2$, under the assumptions of Proposition 4.3 and if $\mathbf{f}_h^l$ has the approximation property specified therein, then the approximate velocities $\mathbf{u}_h^l$ obtained by solving problem (4) satisfy:*

$$\|\mathbf{grad}[\mathbf{u} - \mathbf{u}_h^l]\| \leqslant C_U^l h^{l+1} \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right]. \quad \square \tag{17}$$

Next we have,

Theorem 4.3. *Under the assumptions of Theorem (4.2), the approximate pressures p_h^l obtained by solving problem (4) satisfy:*

$$\|\mathbf{grad}(p - p_h^l)\| \leqslant C_P^l h^l \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right]. \tag{18}$$

Proof. First we note that $\|\mathbf{grad}\, \bar{p}_h^l\| \leqslant \|\mu\Delta\bar{\mathbf{u}}_h^l - \mathbf{grad}\, \bar{p}_h^l\|_h + \mu\|\Delta\bar{\mathbf{u}}_h^l\|_h$.

Taking into account that $\delta = h^2$, from inequality (11), (16) and (17) we derive,

$$\begin{cases} \mu\|\Delta\bar{\mathbf{u}}_h^l\|_h \leqslant \mu C_I \overline{C}^l h^l \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right] \quad \text{and} \\ \|\mu\Delta\bar{\mathbf{u}}_h^l - \mathbf{grad}\, \bar{p}_h^l\|_h \leqslant \overline{C}^l h^l \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right]. \end{cases} \tag{19}$$

(19) and a simple application of the triangle inequality, together with standard estimates lead to (18). □

Theorem 4.3 gives an optimal error estimate for the pressure error measured in the H^1-norm. However an expected $O(h^{l+1})$ error estimate for this error measured in the L^2-norm is lacking. Therefore to complete this study we give:

Theorem 4.4. *Under the assumptions of Theorem (4.2), the approximate pressures p_h^l obtained by solving problem (4) satisfy:*

$$\|p - p_h^l\| \leqslant C_{P0}^l h^{l+1} \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|^2_{l+2,\Gamma_i} \right)^{1/2} \right]. \tag{20}$$

Proof. First of all, using Ladyzhenskaia's condition on the divergence operator [14], we have

$$\|p - p_h^l\| = \sup_{q \in L_0^2(\Omega),\, q \neq 0} \frac{(p - p_h^l, q)}{\|q\|} \leqslant C_M \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega),\, \mathbf{v} \neq \mathbf{0}} \frac{(p - p_h^l, div\, \mathbf{v})}{\|\mathbf{grad}\, \mathbf{v}\|}. \tag{21}$$

On the other hand our choice of δ in (4) together with (1) and (2), imply that $\forall \mathbf{v}_h \in \mathbf{V}_{h0}^l$,

$$(p - p_h^l, div\, \mathbf{v}_h) = -\mu(\mathbf{grad}\, [\mathbf{u}_h^l - \mathbf{u}], \mathbf{grad}\, \mathbf{v}_h) - h^2 \mu(\mu\Delta[\mathbf{u}_h^l - \mathbf{u}] - \mathbf{grad}[p_h^l - p], \Delta\mathbf{v}_h)_h + (\mathbf{f}_h^l - \mathbf{f}, \mathbf{v}_h) + \mu h^2 (\mathbf{f}_h^l - \mathbf{f}, \Delta\mathbf{v}_h)_h, \tag{22}$$

Then from (21) and (22) we easily infer that $\forall \mathbf{v}_h \in \mathbf{V}_{h0}^l$,

$$\|p - p_h^l\| \leqslant C_M \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega),\, \mathbf{v} \neq \mathbf{0}} \left\{ \frac{|(\mathbf{grad}[p_h^l - p], \mathbf{v} - \mathbf{v}_h)| + \mu|(\mathbf{grad}\, [\mathbf{u}_h^l - \mathbf{u}], \mathbf{grad}\, \mathbf{v}_h)|}{\|\mathbf{grad}\, \mathbf{v}\|} + \frac{[\mu C_I h \|\mu\Delta(\mathbf{u}_h^l - \mathbf{u}) - \mathbf{grad}(p_h^l - p)\|_h + (\mu C_I h + C_p)\|\mathbf{f}_h^l - \mathbf{f}\|]\|\mathbf{grad}\, \mathbf{v}_h\|}{\|\mathbf{grad}\, \mathbf{v}\|} \right\}. \tag{23}$$

On the other hand (19) allows us to conclude that

$$\mu\|\Delta[\mathbf{u}_h^l - \mathbf{u}]\|_h \leqslant C_K^l h^l \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^m \|\mathbf{g}\|_{l+2,\Gamma_i}^2 \right)^{1/2} \right]. \tag{24}$$

In order to suitably choose $\mathbf{v}_h$ in (23), we employ a technique similar to the one in the beginning of the proof of Theorem 4.1. Let $\{\mathbf{v}_n\}_n$ be a sequence of $\mathbf{H}^3(\Omega) \cap \mathbf{H}_0^1(\Omega)$ converging to $\mathbf{v}$ in $\mathbf{H}_0^1(\Omega)$. Taking n such that $|\mathbf{v} - \mathbf{v}_n|_1 \leqslant h\|\mathbf{grad}\, \mathbf{v}\|$, we define $\mathbf{v}_h$ to be the $\mathbf{V}_{h0}^l$-interpolate of $\mathbf{v}_n$. Notice that $\|\mathbf{grad}\, \mathbf{v}_n\| \leqslant 2\|\mathbf{grad}\, \mathbf{v}\|$, and necessarily $\|\mathbf{v}_n - \mathbf{v}_h\| \leqslant C_H h \|\mathbf{grad}\, \mathbf{v}_n\|$ and $\|\mathbf{grad}\, \mathbf{v}_h\| \leqslant C'_H \|\mathbf{grad}\, \mathbf{v}_n\|$ (see e.g. [7]). Therefore we have,

$$\|\mathbf{v} - \mathbf{v}_h\| \leqslant (1 + 2C_H) h \|\mathbf{grad}\, \mathbf{v}\| \text{ and } \|\mathbf{grad}\, \mathbf{v}_h\| \leqslant 2C'_H \|\mathbf{grad}\, \mathbf{v}\|. \tag{25}$$

Recalling our assumptions on $\mathbf{f}_h^l$, and combining (23) with (17), (18), (24) and (25), the result follows. □

To conclude this Section we make a few remarks on the terms involving the velocity boundary datum on the right hand side of the estimates given by Theorems 17, 18 and 20.

A first observation these estimates suggest is that the prescribed boundary velocity $\mathbf{g}$ is being assumed to be half a point more regular than one would expect from the assumption $\mathbf{u} \in \mathbf{H}^{l+2}(\Omega)$. However it should be stressed that such an additional regularity is required only away from the boundary corners, and hence it is not at all unreasonable. Nevertheless if one wishes to stick to the assumption that $\mathbf{g} \in \mathbf{H}^{l+3/2}(\Gamma)$, which is coherent with the assumed regularity of $\mathbf{u}$, optimal estimates can still be obtained, if instead of the interpolate of $\mathbf{g}, \mathbf{g}_h$ is the L^2-projection of $\mathbf{g}$ onto the space of boundary traces of functions in $\mathbf{V}_h^l$. Indeed, according to [11] the following estimate holds for such a $\mathbf{g}_h$:

$$\|\mathbf{g} - \mathbf{g}_h\|_{-1/2,\Gamma} \leqslant C^* h^{l+2} \|\mathbf{g}\|_{l+3/2,\Gamma}.$$

Since the term $([\mathbf{g} - \mathbf{g}_h] \cdot \mathbf{n}, \gamma[q])_\Gamma$ can be rewritten as $< [\mathbf{g} - \mathbf{g}_h] \cdot \mathbf{n}, \gamma[q] >_{1/2,\Gamma}$ where $< \cdot, \cdot >_{1/2,\Gamma}$ represents the duality product $\mathbf{H}^{-1/2}(\Gamma) - \mathbf{H}^{1/2}(\Gamma)$ we have:

$$< [\mathbf{g} - \mathbf{g}_h] \cdot \mathbf{n}, \gamma[q] >_{1/2,\Gamma} \leqslant C_H \|\mathbf{g} - \mathbf{g}_h\|_{-1/2,\Gamma} \|\mathbf{grad}\, q\|, \quad \forall q \in Q_h^l.$$

Then the remainder of the estimates would go in the same way as in the three theorems, but in the final estimates we would have to replace the summation of the norms in $\mathbf{H}^{l+2}(\Gamma_i)$ by $\|\mathbf{g}\|_{l+3/2,\Gamma}$. However it should be noted that the construction of such a $\mathbf{g}_h$ is not as straightforward as the one of the interpolate of $\mathbf{g}$, and hence the latter should be the right choice in practice.

It is also worth commenting on the $O(h^{-2})$ factor before the squared norm of R in Proposition 4.1, which certainly would not appear in any stability inequality that holds for the continuous problem. We recall that this term is related to the boundary datum error estimate in Theorems 17, 18 and 20. First of all such a factor is not so unnatural, since we are neither dealing with a coercive problem nor using any uniform stability result on the pairing $\mathbf{V}_h^l - Q_h^l$. Moreover it does not cause any erosion in the method's order of convergence as one might conjecture. This is because we required half a point more piecewise regularity of the boundary datum $\mathbf{g}$, as pointed out above. Actually this is a small price to pay by our method of analysis, that could be avoided by integrating L^2-norms of boundary traces to the working norm of the velocity–pressure approximation space, similarly to [12]. Notice however that in the latter work a bound on c such that $\delta = ch^2$ had to be enforced. In [13] such terms were purged from the working norm, and instead inf–sup inequalities were employed, but in that work only the case of homogeneous velocity boundary conditions was addressed for both the Franca-Hughes [12] and the Douglas-Wang [10] formulations. We conjecture that by using this type of inequalities we could recover full optimality concerning the regularity of $\mathbf{g}$, without sacrificing the arbitrariness of the constant c.

5. Convergence results applying to the Galerkin formulation

In this Section we derive a priori error estimates for the two-dimensional Z_2/P_1 element in standard Galerkin formulation assuming that criss-cross meshes are employed. The rather large number our numerical experiences we performed so far (cf.

Section 6) corroborates the assertion that the Z_2/P_1 element provides optimally convergent sequences of approximations for any kind of mesh. However in contrast to the Taylor–Hood combination P_2/P_1 that is known to have this property, it is not so easy to give formal proofs of it, because of the crucial issue related to element's uniform stability. The main reason is the fact that mid-point velocity degrees of freedom, heavily used to prove the uniform stability of the Taylor–Hood element, are lacking in the structure of Z_2. In this respect we refer to [3] or [15].

For the sake of simplicity, and without loss of essential results, we assume that $\mathbf{g} = \mathbf{0}$. Furthermore we confine our convergence analysis to a particular kind of triangulations of Ω. More specifically we assume that the meshes are of the *criss-cross* type. This means that the partition $\mathcal{T}_h$ is generated by subdividing into four triangles each element Q of a first partition $\mathcal{Q}_h$ of Ω into convex quadrilaterals, by means of the two diagonals of Q, as illustrated in Fig. 2.

Notice that, since the final partitions $\mathcal{T}_h$ are assumed to form a quasi-uniform family, this must also be the case of the corresponding family of partitions $\mathcal{Q}_h$ (cf. [7]).

It is well-known that problem (4) has a unique solution for $\delta = 0$ if and only if the following *inf–sup* condition holds (see e.g. [6]):

$$\begin{cases} \exists \beta_h^l > 0 \text{ such that } \forall q \in Q_h^l, \\ \sup\limits_{\mathbf{v} \in \mathbf{V}_{h0}^l,\ \mathbf{v} \neq \mathbf{0}} \frac{(div\ \mathbf{v}, q)}{\|\mathbf{grad}\ \mathbf{v}\|} \geqslant \beta_h^l \|q\|. \end{cases} \tag{26}$$

According to the celebrated theory due to Stenberg [23], condition (26) holds with a constant β_h^l independent of h if a certain macro-element condition is satisfied. This condition is stated in general terms in [23]. For better guidance we recall it here, restricting ourselves to $l = 1$ and to the specific case where the macro-elements into which Ω is partitioned are quadrilaterals $Q \in \mathcal{Q}_h$.

Theorem 5.1 (*adapted from [23]*). *Assume that the quadrilaterals in $\mathcal{Q}_h$ are equivalent to a reference unit square $\widehat{Q}$ in the sense that $\forall Q \in \mathcal{Q}_h$ there is a mapping $\mathcal{F}_Q : \widehat{Q} \longrightarrow Q$ such that,*

(i) $\mathcal{F}_Q$ is continuous and one-to-one;
(ii) $\mathcal{F}_Q(\widehat{Q}) = Q$;
(iii) The triangles T_i in Q satisfy $\mathcal{F}_Q(\widehat{T}_i) = T_i$, for $i = 1, 2, 3, 4$;
(iv) $\mathcal{F}_Q = \mathcal{F}_{T_i} \circ \mathcal{F}_{\widehat{T}_i}^{-1}, i = 1, 2, 3, 4$, where $\mathcal{F}_{\widehat{T}_i}$ and $\mathcal{F}_{T_i}$ are the affine mappings from the reference triangle with vertices $(0,0), (1,0), (0,1)$ onto $\widehat{T}_i$ and T_i, respectively, $i = 1, 2, 3, 4$.

Denoting by $(\cdot,\cdot)_Q$ the inner product of $L^2(Q)$, for each $Q \in \mathcal{Q}_h$ further assume that the space $N_Q := \{p \in P_Q \mid (div\,\mathbf{v}, p)_Q = 0\ \forall \mathbf{v} \in \mathbf{V}_Q\}$ consists of functions that are constant in Q, with $P_Q := \{p \in C^0(\overline{Q}) \mid p_{|T_i} \in P_1, i = 1, 2, 3, 4\}, \mathbf{V}_Q$ being the space of restrictions $\mathbf{v}_Q$ to Q of fields in $\mathbf{V}_h^1$ such that $\mathbf{v}_Q \in [H_0^1(Q)]^2$. Then the condition (26) holds for $l = 1$ with β_h^1 independent of h. □

Since the four geometric conditions enumerated in Theorem 5.1 are clearly satisfied, all that is left to do is proving that N_Q satisfies the other condition stated therein. Let then $Q \in \mathcal{Q}_h$ and V_Q be the space of restrictions of functions in V_h^1 that vanish on the boundary of Q. We recall that P_Q is the space of continuous functions in Q that are linear in each triangle of $\mathcal{T}_h$

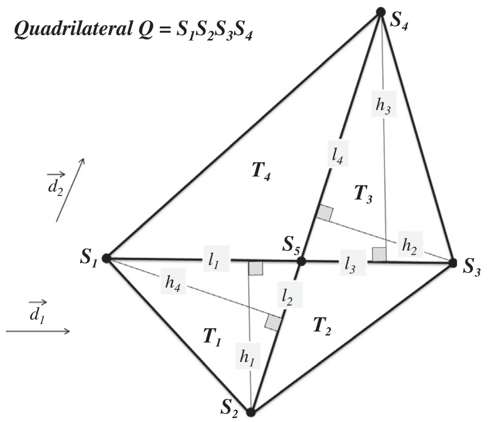

Fig. 2. Assembly of four triangles of $\mathcal{T}_h$ into a convex quadrilateral Q and pertaining lengths.

contained in Q, referred to as $T_0, T_1, T_2, T_3, T_4, T_5$, with $T_5 = T_1$ and $T_0 = T_4$, as indicated in Fig. 2. Let also l_i be the length of segment S_iS_5 for $i = 1,2,3,4$ and h_i (resp. h_{i+2}) be the height of triangles T_i and T_{i+1} (resp. T_{i+2} and T_{i+3}) with respect to the direction of S_iS_5, for $i = 1,2$, with $h_0 = h_4$.

Referring to Fig. 2 we note that every $q \in P_Q$ is a function of the form $\sum_{i=1}^{5} q_i\eta_i$, where η_i is the function in P_Q satisfying $\eta_i(S_j) = \delta_{ij}$, that is, $q_i = q(S_i), i = 1,\dots,5$. As for V_Q, let us denote by $\vec{d}_1$ and $\vec{d}_2$ the unit vectors of edges S_1S_3 and S_2S_4 respectively oriented as indicated, by ϕ_Q the function in V_Q whose gradient vanishes at S_5 and by ϕ_Q^j the function in V_Q such that $\phi_Q^j(S_5) = 0$, and $\frac{\partial\phi_Q^j}{\partial d_k}(S_5) = \delta_{jk}$ for $j,k \in \{1,2\}$, the meaning of the partial derivative being obvious. Notice that $\mathbf{V}_Q$ is spanned by six basis fields $\mathbf{v}_Q^i, i = 1,2$, and $\mathbf{v}_Q^{jk}, j,k \in \{1,2\}$, where:

- $\mathbf{v}_Q^i = \phi_Q \vec{d}_i, i = 1,2;$
- $\mathbf{v}_Q^{jk} = \phi_Q^j \vec{d}_k, j,k \in \{1,2\}.$

Next we prove

Proposition 5.1. *If $q \in P_Q$ and $(div\,\mathbf{v}, q)_Q = 0\ \forall \mathbf{v} \in \mathbf{V}_Q$ then q is constant in Q.*

Proof. First we note that $(div\,\mathbf{v}, q)_Q = -(\mathbf{v}, \mathbf{grad}\, q)_Q\ \forall \mathbf{v} \in \mathbf{V}_Q$ and $\forall q \in P_Q$.

Let $q \in P_Q$ and $(\mathbf{v}, \mathbf{grad}\, q)_Q = 0\ \forall \mathbf{v} \in \mathbf{V}_Q$. From $(\mathbf{v}_Q^i, \mathbf{grad}\, q)_Q = 0\ (i = 1,2)$ one gets

$$\frac{q_5 - q_i}{l_i}\int_{T_i \cup T_{i-1}} \phi_Q\, dx_1 dx_2 + \frac{q_{i+2} - q_5}{l_{i+2}}\int_{T_{i+2} \cup T_{i+1}} \phi_Q\, dx_1 dx_2 = 0.$$

Noticing that in every triangle T_m, ϕ_Q is a function of the form φ_i defined in Section 2, well-known integration formulae in a triangle (cf. [27]) yield $\int_{T_m} \phi_Q\, dx_1dx_2 = area(T_m)/3, m = 1,2,3,4$ (cf. [27]). Then, since $area(T_i) = l_ih_i/2 = l_{i+1}h_{i-1}/2$, after straightforward calculations we obtain,

$$(h_i + h_{i+2})(q_{i+2} - q_i) = 0, \text{ for } i = 1,2, \text{ that is, } q_1 = q_3 \text{ and } q_2 = q_4. \tag{27}$$

Considering now the function ϕ_Q^1, we note that $(\mathbf{v}_Q^{11}, \mathbf{grad}\, q)_Q = 0$ implies that

$$\frac{q_5 - q_1}{l_1}\int_{T_1 \cup T_4} \phi_Q^1\, dx_1 dx_2 + \frac{q_3 - q_5}{l_3}\int_{T_2 \cup T_3} \phi_Q^1\, dx_1 dx_2 = 0. \tag{28}$$

Notice that the restriction of ϕ_Q^1 to each triangle T_m is a function of the type ζ_{ij} defined in Section 2 multiplied by either $-l_1$ for T_1 and T_4 or by l_3 for T_2 and T_3. Hence its integral in T_1 and T_4 is equal to $-l_1$ area $(T_m)/24$ for $m = 1$ and $m = 4$, and to l_3 area $(T_m)/24$ for $m = 2$ and $m = 3$. It follows that (28) yields,

$$(q_1 - q_5)\,\text{area}\,(T_1 \cup T_4) + (q_3 - q_5)\,\text{area}\,(T_2 \cup T_3) = 0. \tag{29}$$

Then from (29) and (27) it immediately follows that $q_5 = q_3 = q_1$.

Now using symmetry we derive from $(\mathbf{v}_Q^{22}, \mathbf{grad}\, q)_Q = 0$ the relation

$$(q_2 - q_5)\,\text{area}\,(T_1 \cup T_2) + (q_4 - q_5)\,\text{area}\,(T_3 \cup T_4) = 0. \tag{30}$$

Finally combining (30) with (27), we derive $q_5 = q_4 = q_2$, and the result follows. □

Remark 5.1. It can be easily checked that the relations $(\mathbf{v}_Q^{12}, \mathbf{grad}\, q)_Q = 0$ and $(\mathbf{v}_Q^{21}, \mathbf{grad}\, q)_Q = 0$ do not bring about any additional information. For instance taking $\mathbf{v}_Q^{12}$ we derive $(q_5 - q_2)(l_3h_2 - l_1h_4) + (q_4 - q_5)(l_3h_2 - l_1h_4) = 0$, which is trivially fulfilled taking into account (27). □

As a consequence of Proposition 5.1 we can state,

Theorem 5.2. *Let Ω be a polygonal domain. If the triangulation $\mathcal{T}_h$ is of the criss-cross type, there exist a unique velocity field $\mathbf{u}_h^1 \in \mathbf{V}_h^1 \cap [H_0^1(\Omega)]^2$ and a unique pressure $p_h^1 \in Q_h^1$ that solve (4) with $\delta = 0$. Furthermore, provided $\mathbf{u} \in \mathbf{H}^3(\Omega)$ and $p \in H^2(\Omega)$, these fields satisfy for a suitable constant C_G,*

$$\|\mathbf{grad}\,(\mathbf{u} - \mathbf{u}_h^1)\| + \|p - p_h^1\| \leqslant C_G h^2\{|\mathbf{u}|_3 + |p|_2\}. \tag{31}$$

Proof. This Theorem is an immediate consequence of the validity of (26) with β_h^1 bounded away from zero by a constant $\beta^* > 0$ for every $\mathcal{T}_h$, thanks to Proposition 5.1 and to the macro-element criterion [23], together with standard results in [7,6]. □

Remark 5.2. It seems difficult to extend the above analysis to the case of other types of triangulations. Nevertheless, using arguments similar to [2], it is rather easy to prove that the method is stable for any triangulation, provided the space Z_2 is replaced by Z_3 in all the triangles, whose closure has a non empty intersection with Γ. Actually, even if Z_2 is maintained for such triangles, there are several ways of obtaining theoretically stable versions of the two-dimensional Z_2/P_1 method in Galerkin formulation for arbitrary triangular meshes. For instance this can be achieved by performing minor modifications in the definition of $\mathbf{V}_h^1$ in the neighborhood of Γ. However as we will see in the next Section, all this seems unnecessary, since this finite element method in Galerkin formulation behaves like a stable and convergent one, irrespective of the type of meshes being used. □

6. Assessment of the Zienkiewicz triangle to represent the velocity in Stokes flow

We performed several numerical experiments with the two-dimensional elements. Most of the corresponding results are reported below.

6.1. The Z_2/P_1 element

A stabilized (Galerkin-Least-Squares) formulation with the Z_2/P_1 elements and a stabilization parameter $\delta = Ch^2$, of the Stokes problem with arbitrarily prescribed velocities on the boundary, has been proved in Section 4 to be convergent to the exact solution with optimal order and no restriction on C other than $C > 0$. In Section 5 it has also been demonstrated that the same property holds for the standard Galerkin formulation, if families of meshes of the criss-cross type are employed. Our first goal here is to assess the performance of the Z_2/P_1 element as compared to the classical P_2/P_1 known as the Taylor–Hood element, in terms of accuracy (though their asymptotical orders, being optimal, must coincide). Our second goal is to study the dependence of the numerical solution on the parameter δ and, perhaps more interestingly, to numerically assess whether the plain Galerkin formulation is convergent for the Z_2/P_1 element for other kinds of mesh families too. In fact, our results suggest that this element uniformly satisfies the *inf–sup* condition (26) for any regular family of meshes, though a mathematical proof of such a property is unavailable.

A manufactured solution was used for the numerical tests. It is defined by the stream function:

$$\psi(x_1,x_2) = x_1^2(1-x_1)^2 x_2^2(1-x_2)^2 = g(x_1)g(x_2), \tag{32}$$

where $g(\xi) = \xi^2(1-\xi)^2$. The corresponding velocity field is

$$u_1(x_1,x_2) = g(x_1)g'(x_2), \quad u_2(x_1,x_2) = -g'(x_1)g(x_2), \tag{33}$$

which vanishes on the boundary of the domain $\Omega = (0,1) \times (0,1)$. Choosing the viscosity $\mu = 100$ and pressure field as

$$p(x_1,x_2) = x_1 - x_1^2, \tag{34}$$

one can compute the force field $\mathbf{f}$ that makes $(\mathbf{u}, p)$ a solution of the Stokes flow inside Ω, with rigid-wall boundary conditions ($\mathbf{u} = \mathbf{0}$ on $\partial\Omega$), from

$$\mathbf{f} = -\mu\Delta\mathbf{u} + \mathbf{grad}p. \tag{35}$$

Four uniform criss-cross meshes and four quasi-uniform unstructured meshes were built for the assessment. The criss-cross meshes consist of $3 \times 3, 6 \times 6, 12 \times 12$ and 24×24 squares, each divided into four triangles. The corresponding mesh sizes are $h = 0.333$, $0.166, 0.083$ and 0.041, respectively. In turn, the data corresponding to the unstructured meshes are shown in Table 1. The mesh size h corresponds to the largest edge length in the mesh.

The stabilization parameter δ is written as

$$\delta = c_\delta \frac{h^2}{10\mu}$$

and we consider $c_\delta = 0$ (Galerkin formulation) and $c_\delta = 1$ (stabilized formulation). Also incorporated in the comparison is a plain-vanilla P_2/P_1 Galerkin finite element code, with the meshes obtained by inserting additional nodes at the midpoints of each edge of the Zienkiewicz elements.

Table 1
Information on the unstructured meshes used.

Mesh	# Triangles	# Vertices	h	# Unknowns	
				P_2/P_1	Z_2/P_1
M1	40	29	0.290	223	203
M2	180	107	0.150	893	749
M3	690	378	0.077	3268	2646
M4	2610	1370	0.039	12068	9590

Table 1 reports the number of unknowns for each mesh and for both the Z_2/P_1 and P_2/P_1 elements. The former is more economical, with asymptotically 7/9 the unknown count of the latter.

The computed errors are $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ (Fig. 3(a)), $\|p-p_h\|_{L^2(\Omega)}$ (Fig. 3(b)), $\|\mathbf{grad}\,\mathbf{u}-\mathbf{grad}\,\mathbf{u}_h\|_{L^2(\Omega)}$ (Fig. 4(a)), $\|\mathbf{grad}\,p-\mathbf{grad}\,p_h\|_{L^2(\Omega)}$ (Fig. 4(b)), $\|div\,\mathbf{u}-div\,\mathbf{u}_h\|_{L^2(\Omega)}$ (Fig. 5)) and $\|D^2\mathbf{u}-D^2\mathbf{u}_h\|_h$ (Fig. 5(b)). The figures plot the errors as functions of h for the Z_2/P_1 element in criss-cross meshes for $c_\delta=0$ and in unstructured meshes for both $c_\delta=0$ and $c_\delta=1$, together with similar plots for the P_2/P_1 element for $c_\delta=0$.

In each figure an additional line is drawn indicating the optimal order of convergence. It can be seen that all the schemes, *even the Galerkin method with the proposed Z_2/P_1 element in unstructured meshes*, are optimally convergent. Further, this latter element turns out to be the most accurate of the three, without any sign of instability as $h\to 0$.

The last test performed investigates the best choice for the stabilization constant c_δ, which has been taken as either zero or one up to now. For this purpose, we adopt a norm that combines the velocity and pressure fields in a unit-consistent way, i.e.,

$$\|(\mathbf{u},p)\|_\mu \stackrel{\text{def}}{=} \left(\mu\|\mathbf{grad}\,\mathbf{u}\|^2_{L^2(\Omega)}+\frac{1}{\mu}\|p\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{36}$$

In Fig. 6 we plot the error (i.e., $\|(\mathbf{u},p)-(\mathbf{u}_h,p_h)\|_\mu$) as a function of c_δ, for the four unstructured meshes M1-M4. One observes that there is no change of behavior of the method as c_δ approaches zero, thus strongly suggesting that the Z_2/P_1 element is stable in the Galerkin formulation (i.e.; suggesting that it satisfies the inf–sup condition in general meshes). As a function of c_δ, the error is essentially constant from $c_\delta=0$ up to $c_\delta\simeq 1$, at which point the effect of the stabilization becomes significant. For values $c_\delta\gg 1$ an important loss of accuracy is observed, thus justifying the choice $c_\delta=1$ adopted previously. However, from a practical viewpoint it is the Galerkin formulation that should be favored ($c_\delta=0$), since it involves less computations without any loss in accuracy.

6.2. A failed attempt at further reducing the number of degrees of freedom

Since an advantage of the Z_2/P_1 element is the smaller number of unknowns, a natural question that arises is whether this number can be further reduced without loosing approximation capabilities. The nature of the nodal unknowns provides further motivation: The velocity unknowns correspond to the value of $\mathbf{u}_h$ and of $\mathbf{grad}\,\mathbf{u}_h$ at each node, totalling six unknowns. Since the trace of $\mathbf{grad}\,\mathbf{u}$ (i.e.; $div\,\mathbf{u}$) vanishes identically, the linear combination of unknowns that determines $div\,\mathbf{u}_h$ at the nodes does not contribute to the interpolation accuracy. Let us define the subspace of Z_2 consisting of vector fields with zero divergence at the nodes,

$$Z_2^{\text{div}}=\{\mathbf{w}_h\in Z_2 | div\,\mathbf{w}_h(S)=0,\ \forall S \text{ node of } \mathcal{T}_h\}. \tag{37}$$

The Z_2^{div}/P_1 element has five velocity unknowns per node, thus leading to 6/7 times the number of unknowns of the Z_2/P_1 element and (asymptotically) 2/3 times that of the P_2/P_1 element. Unfortunately, the Z_2^{div}/P_1 element does not converge in the Galerkin formulation. Stabilizing it by taking $c_\delta=1$ convergence is attained, as shown in Fig. 7 where for comparison purposes the error of the stabilized Z_2/P_1 element is also plotted. Though the figure shows evidence of correct convergence orders and small differences in the velocity error, the Z_2^{div}/P_1 element exhibits much larger pressure errors than the Z_2/P_1 element and thus its convenience is questionable.

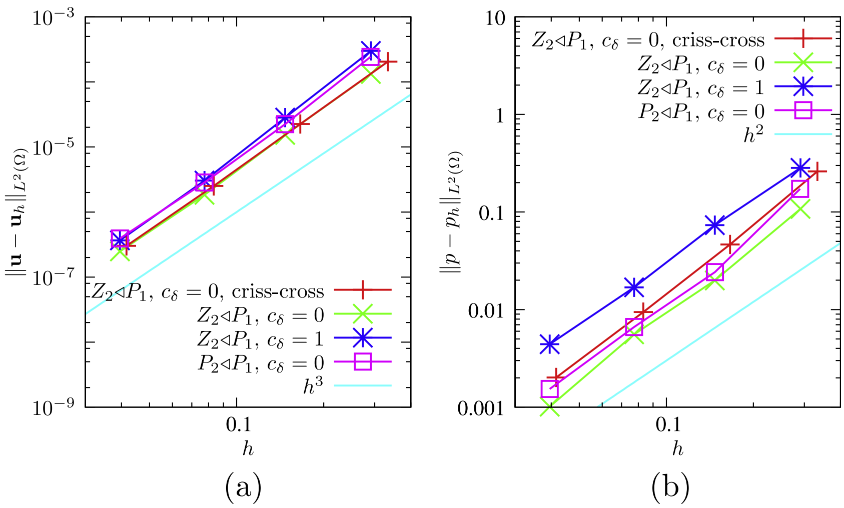

Fig. 3. Comparison of errors of (a) $\mathbf{u}_h$ and of (b) p_h in $L^2(\Omega)$.

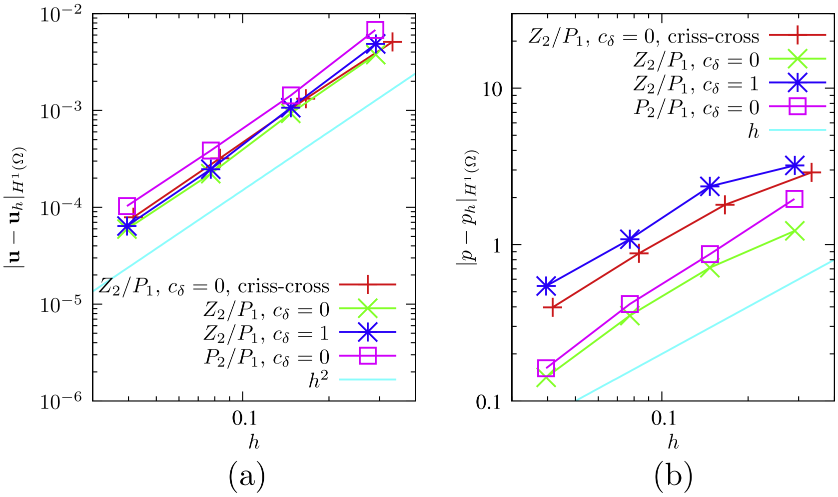

Fig. 4. Comparison of errors of (a) **grad** $\mathbf{u}_h$ and of (b) **grad** p_h in $L^2(\Omega)$.

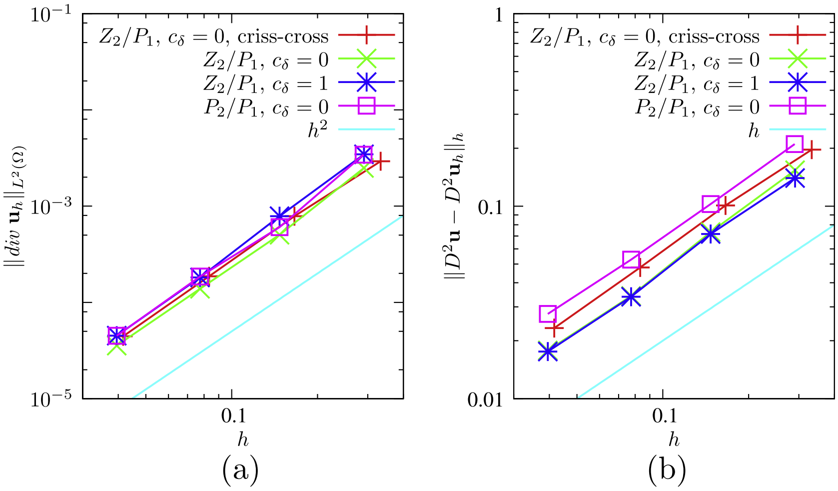

Fig. 5. Comparison of errors of (a) $div\mathbf{u}_h$ in $L^2(\Omega)$ and of (b) $D^2\,\mathbf{u}_h$ in the norm $\|\cdot\|_h = \left(\sum_T \|\cdot\|^2_{L^2(T)}\right)^{\frac{1}{2}}$.

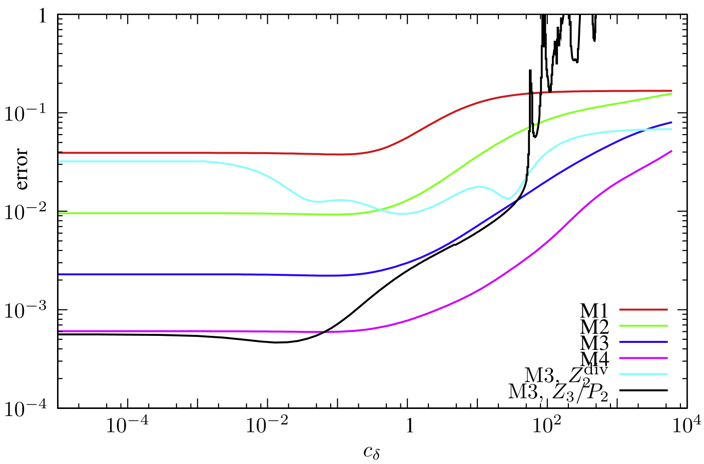

Fig. 6. Error $\|(\mathbf{u}, p) - (\mathbf{u}_h, p_h)\|_\mu$ as a function of c_δ for the four unstructured meshes considered in the study. Also plotted is the curve corresponding to the Z_2^{div}/P_1 element (cf. Section 6.2) and and to the Z_3/P_2 element (cf. Section 6.3), both computed on mesh M3.

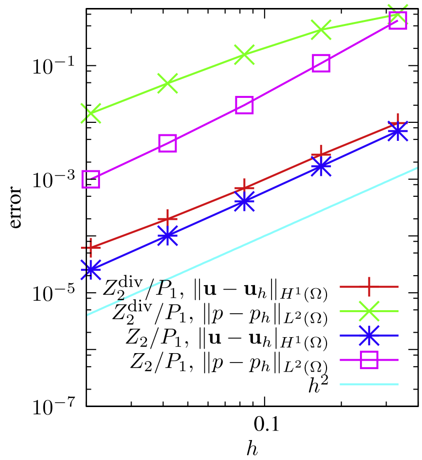

Fig. 7. Convergence behavior of the $Z_2^{\rm div}/P_1$ element in the stabilized formulation, as compared to the Z_2/P_1 one. The results correspond to criss-cross meshes.

Since the unsatisfactory behavior of the element $Z_2^{\rm div}/P_1$ could come from an inadequate choice of the stabilization constant, in Fig. 6 we plot the error obtained on mesh M3 as a function of c_δ. One observes that the minimum error roughly corresponds to $c_\delta = 1$ but, contrary to what occurs with the Z_2/P_1 element, the error grows significantly as c_δ is reduced. For $c_\delta = 10^{-3}$, for example, the error is three times that obtained with $c_\delta = 1$. This is seen as an empirical evidence of Galerkin formulation's lack of stability.

Remark 6.1. The space $Z_2^{\rm div}$ is remarkable in that any field $\mathbf{w}_h \in Z_2^{\rm div}$, though not being divergence-free everywhere inside Ω, is divergence-free *at the nodes*. If a finite element space consists of divergence-free functions the pressure can be eliminated as unknown and an elliptic, strongly coercive variational problem solved for the velocity field alone. To see whether the space $Z_2^{\rm div}$ could serve as an "approximately divergence-free space", we implemented the pressure-free formulation "Find $\mathbf{z}_h \in Z_2^{\rm div}$ such that for all $\mathbf{v} \in Z_2^{\rm div}$ (with the appropriate boundary conditions)

$$\delta(\mu\Delta\mathbf{z}_h, \mu\Delta\mathbf{v}) + \mu(\mathbf{grad}\,\mathbf{z}_h, \mathbf{grad}\,\mathbf{v}) = -\delta(\mathbf{f}_h, \mu\Delta\mathbf{v}) + (\mathbf{f}_h, \mathbf{v}). \tag{38}$$

Unfortunately, the field $\mathbf{z}_h$ obtained from this formulation does not converge to the exact solution. □

6.3. The Z_3/P_2 element

The same manufactured solution of the previous section was used to assess the convergence of the Z_3/P_2 element on the unstructured meshes M1-M4 of Table 1. A stable behavior was observed for any choice of the stabilization parameter c_δ, suggesting that the Z_3/P_2 element satisfies the *inf–sup* condition. In Fig. 6 we have plotted the error (in the norm $\|\cdot\|_\mu$) obtained on mesh M3 as a function of c_δ. The error-minimizing value in this norm appears to be $c_\delta \simeq 10^{-2}$ instead of 1, but as happens for the Z_2/P_1 element the gain in accuracy with respect to the Galerkin formulation ($c_\delta = 0$) is only marginal. Another consideration to favor the Galerkin formulation is the steep growth of the error when c_δ is very large (about 100 for mesh M3), since the meaning of "very large" can be mesh or problem dependent. Notice also that the Z_3/P_2 element on mesh M3, with just 3713 unknowns, yields a smaller error than both the Z_2/P_1 and the P_2/P_1 elements on mesh M4, involving 9590 and 12068 unknowns, respectively.

The Galerkin formulation of the Z_3/P_2 element is of course not covered by the theoretical results of the previous sections and a convergence assessment is in order. Numerical results of the velocity error in the $H^1(\Omega)$-norm and of the pressure error in the $L^2(\Omega)$-norm are plotted in Fig. 8. Optimal ($O(h^3)$) accuracy is observed, which suggests that the inf–sup constant of this element is indeed independent of h (uniform div-stability).

6.4. Problems with limited regularity

The previous numerical experiments have all considered the analytical streamfunction (32). It is however important to evaluate the performance of the proposed elements in problems with limited regularity, as they are most frequent in applications.

For this purpose, consider the following streamfunction

$$\psi(x_1, x_2) = x_1^s(1-x_1)^s x_2^s(1-x_2)^s, \tag{39}$$

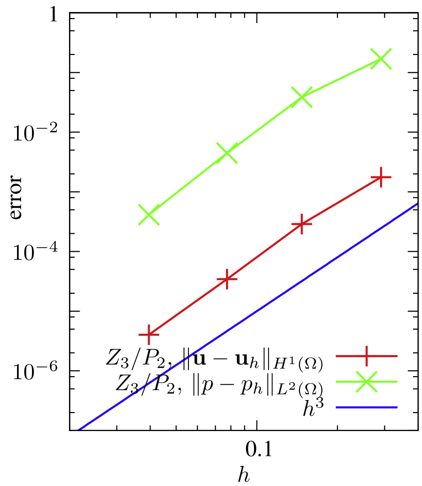

Fig. 8. Convergence behavior of the Z_3/P_2 element in the Galerkin formulation. The results correspond to the unstructured meshes M1-M4.

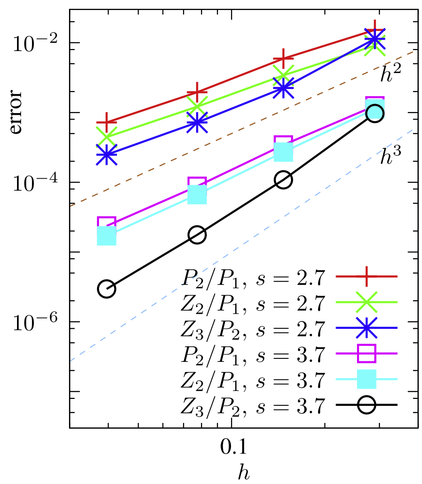

Fig. 9. Errors in the norm $\|\cdot\|_\mu$ of the $P_2/P_1, Z_2/P_1$ and Z_3/P_2 elements, all in the Galerkin formulation, for the limited-regularity problem defined by the streamfunction (39). Shown are two cases, corresponding to exponents $s = 2.7$ and $s = 3.7$.

from which the exact velocity field can be obtained. The force field $\mathbf{f}$ is obtained as before, taking $\mu = 100$ and

$$p(x_1, x_2) = x_1 - x_1^3 + x_2 - x_2^3.$$

When s is not a natural number the resulting velocity field has only a finite number of bounded derivatives in $\Omega = (0,1) \times (0,1)$. In fact, it is easily shown that for $\mathbf{u}$ to belong to $H^m(\Omega)$ the exponent s must be greater than $m + 1/2$.

Numerical experiments were carried out to compare the performance of the $P_2/P_1, Z_2/P_1$ and Z_3/P_2 elements in this problem. The error was measured in the norm $\|\cdot\|_\mu$ for the unstructured meshes M1-M4, taking $s = 2.7$ and 3.7 (and thus $\mathbf{u}$ belonging to $H^2(\Omega)$ and $H^3(\Omega)$, respectively, but not more).

The results are shown in Fig. 9. For the case $s = 2.7$ the expected convergence rate is $O(h^p)$ with $p < 2$, since $\mathbf{u} \notin H^3(\Omega)$. This is confirmed by the results obtained with all three elements, which exhibit $p \simeq 1.7$. On the other hand here again one observes that Z_2/P_1 and Z_3/P_2 are more accurate than the P_2/P_1 element.

For the case $s = 3.7$ the expected convergence rate is $p = 2$ for P_2/P_1 and Z_2/P_1, since $\mathbf{u} \in H^3(\Omega)$, and $p < 3$ for Z_3/P_2, because $\mathbf{u} \notin H^4(\Omega)$. These orders are confirmed by the experimental results, with the Z_3/P_2 almost achieving third order and the Z_2/P_1 again exhibiting more accuracy than the P_2/P_1 element.

7. Miscellaneous remarks

It seems important to stress some merits of the Hermite elements studied in this paper. First of all we can state that they have an a priori advantage over Lagrange elements of the same order in terms of cost. More specifically, one may compare second order method Z_2/P_1 with the Taylor–Hood element P_2/P_1 [17] based on a classical Lagrange quadratic representation of the velocity and a continuous piecewise linear representation of the pressure, which is also a second order element in the $H^1 \times L^2$-norm. A simple count shows that in the two-dimensional case, on the same mesh, the ratio between the number of velocity degrees of freedom of our second order element and the one of the Taylor–Hood element is roughly 7/9. In the three-dimensional case this ratio becomes even more favorable, for it is reduced to about one half. Concerning the third order element Z_3/P_2, a fair comparison is to be made with the P_3/P_2 element, i.e., Lagrange cubics for the velocity and quadratics for the pressure. In this case the above specified ratios are ca. 0.6 in both two- and three-dimensions. Notice however that in the two-dimensional case we may use inner node static condensation for both third order elements being compared, and in this case the velocity degree of freedom ratio is reduced to a little more than 0.4.

It is interesting to point out that the Zienkiewicz triangle had been used by other authors too, in order to simulate viscous incompressible flow problems (cf. [8]). It turns out that the numerical results they showed are as encouraging as ours.

Another feature of Hermite pseudo-C^1 elements like those we studied here, is the fact that second order derivatives can be computed element by element with acceptable accuracy, directly from the numerical velocity field. This can be achieved by first interpolating the velocity gradients continuously at the vertices of the mesh, using continuous piecewise linear functions, which can be differentiated in each element. Actually the authors intend to further exploit these Hermite methods in the near future, in the simulation of flow on curved manifolds [26], in which an accurate determination of velocity derivatives is a must.

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