A NEW CAVITATION MODEL IN LUBRICATION: THE CASE OF TWO–ZONE CAVITATION

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ABSTRACT. A generalization of the Elrod-Adams model of cavitation in lubricated devices is proposed, such that the translation velocity V for the saturation field θ can be given any value between $S/2$ and S. The lack of uniqueness of the classical model when $V \neq S/2$ is explained and a suitable supplementary condition is proposed to fix this issue. The new model is rigorously analyzed, though in the simplified mathematical setting of a one-dimensional problem with a single pressurized region. The main result states the existence of a unique solution globally in time, unless of course the cavitation boundary leaves the domain or disappears. A few preliminary numerical examples are included to illustrate the model.

1. INTRODUCTION

Cavitation in lubricated bearings is a classic subject in tribological modeling[12]. It is characterized as the presence of a gas or vapour phase within the liquid lubricant, in such a way that the pressure is essentially constant in those regions where the gas and the lubricant coexist. This may be due to the gas coming out of solution from the lubricant, or due to the direct exposure to the atmosphere and consequent exchange of gas.

A thorough discussion of the cavitation phenomenon in bearings can be found in the review by Dowson & Taylor [13]. Braun & Hannon [7] provide a recent update, with emphasis on the modeling aspects. Essentially, the most popular model for numerical predictions is due to Elrod & Adams [14], with further algorithmic improvements by Vijayaraghavan & Keith [26], Kumar & Booker [19], Ausas *et al* [3], among others. The model involves two unknowns, namely the pressure p and a saturation variable θ which represents the liquid content at each location.

The Elrod-Adams or modified $p-\theta$ models are accurate enough for most of the lubricated devices in use. However, the advent of surface-finishing techniques that allow for the control of the microscopic geometry of the surfaces has led to new challenges in tribological modeling [15, 25, 27]. Industry pushes towards the assessment of surface textures, making theoretical/numerical models [8, 16, 18, 23, 24, 20, 28, 29] and optimization results thereof [9, 10, 11] the subject of intensive research.

In textured surfaces cavitation happens both at the macroscopic scale and at the scale of a single texture cell. The cavitation model is crucial for the accurate determination of the cavitation boundaries which in turn determine the load capacity, friction and other properties of the bearing[1]. Deficiencies of cavitation models are blatantly put in evidence by textured surfaces, especially when the textures are in relative motion.

The Elrod-Adams model, in particular, is not flexible enough for some problems. This model translates the lubricant content variable θ in the cavitated region at speed $V = S/2$, where S is the speed of the moving surface. This is realistic if lubricant and gas are in a complex, highly mixed configuration (streamers), but if the lubricant lies mainly on the moving surface the speed V should take the value aS, with $a \simeq 1$. This situation is depicted in Fig. 1, from which it is intuitive to infer that the velocity of the fluid for x far from the pressurized region will be S, and not $S/2$. The frame of reference moves at speed $-S$ with the piston ring and thus the oil that is sitting on the cylinder must approach the ring at speed $V = S$. This defficiency of the Elrod-Adams model, which in a microtextured case leads to unrealistic motion of the film in each microcavitation region, was pointed out by Organisciak [22]. He also found that a naive modification of the model (simply changing the θ-translation speed whenever $\theta < 1$) led to the loss of uniqueness of the solution. In the simulations, wild oscillations of the θ field develop at rupture boundaries [2].

In the case of two–zone cavitation, namely the one–dimensional setting of Fig. 1 consisting of a single connected pressurized region, the Elrod-Adams model is generalized here in such a way that the translation velocity V for θ can be given any value between $S/2$ and S. It is shown that, if $V \neq S/2$, the solution is not unique because there appears a finite interval $[F(\beta'(t)), 1]$ of possible values of the lubricant content variable θ at the *rupture* boundary β, when the latter moves with speed $\beta'(t)$. All these values are compatible with mass conservation and with a positive

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pressure field in the pressurized region (the lower limit, $\theta = F(\beta'(t))$, corresponds to $\nabla p = 0$ at rupture). In the Elrod-Adams model ($V = S/2$) this indeterminacy does not exist, since $F(\beta'(t)) = 1$. To fully determine the solution with $V \neq S/2$ a supplementary condition on θ at rupture boundaries is needed. We propose one possible such condition (which could be called a rupture model), which essentially consists of taking θ equal to some constant value λ, unless $\lambda < F(\beta'(t))$, in which case the latter value is taken. We are certain that physically more realistic rupture models can be developed by making use of detailed experiments or multiphase simulations. The underlying mathematics, however, are already present in the simple model proposed here and to which all the analysis is devoted. Though the proof is quite technical, the main result essentially states that, after adding the rupture condition to the cavitation model with $V \neq S/2$, the problem is well-posed. More specifically, the existence of a unique solution is proved, which holds either for infinite time, or until one of the natural limits of the simulation is reached: (a) the cavitation boundary leaves the simulation domain, or (b) the pressurized region collapses to a point and disappears. A few preliminary numerical examples are included to illustrate the model, and further assessments can be found in a companion article [4].

2. Statement of the problem

2.1. **Physical problem.** We consider two surfaces which are in relative motion, with a small clearance between them which is given by some continuously differentiable function $h(x) \geq h_{\min} > 0$. For simplicity, we assume that the lower surface is planar and moves with constant speed $S > 0$ along x, while the upper one is fixed. The domain in which the two surfaces are in proximity is normalized to $\Omega =]0, 1[$.

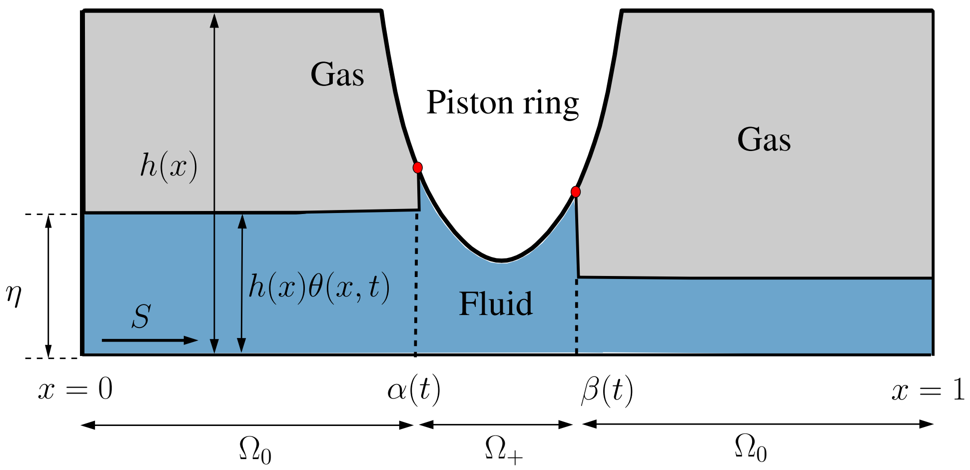

FIGURE 1. Representation of the problem's domain and main variables. Notice that the fluid film (with thickness $h\,\theta$) is drawn as attached to the lower surface.

An incompressible lubricant fluid flows between the two surfaces, occupying a fraction $\theta(x, t)$ of the clearance ($0 \leq \theta \leq 1$). At each instant t, the domain Ω can be divided into two regions (see Fig. 1):

– The region $\Omega_+(t)$ where the fluid completely fills the gap ($\theta = 1$), which is called the *pressurized* or *active* region. There, a *non-negative* pressure field $p(x, t)$ develops that, under suitable conditions, obeys the celebrated Reynolds lubrication equation (here in non–dimensional form)

$$\partial_x \left(h^3 \partial_x p\right) = \frac{S}{2} \partial_x h \tag{2.1}$$

– The region $\Omega_0(t)$, where the fluid film is incomplete ($\theta < 1$), which is known as the *cavitated* zone. Inside it the pressure is assumed to be zero and the lubricant content at each point obeys a pure transport equation

$$\partial_t(\theta\, h) + V\, \partial_x(\theta\, h) = 0 \tag{2.2}$$

where V is the constant transport velocity which will be defined later.

The boundary between Ω_+ and Ω_0 is the so-called *cavitation boundary* Σ and is an unknown of the problem. The setting is developed so as to model a piston ring sliding against the cylinder liner, with the frame of reference moving with the ring. The ring (upper surface in Fig. 1) occupies the center of the domain, so that the pressurized region is assumed to lie between $x = \alpha(t)$ and $x = \beta(t)$, with $0 < \alpha(t) < \beta(t) < 1$. Notice that the cavitated region is, thus, $\Omega_0(t) = (0, \alpha(t)) \cup (\beta(t), 1)$, with $\alpha(t)$ and $\beta(t)$ unknowns. The other unknowns are the pressure p in Ω_+, with boundary condition

$$p = 0 \qquad \text{on } \Sigma \tag{2.3}$$

and θ in Ω_0, with initial condition $\theta(x, t = 0) = \theta_0(x)$. A lubricant film of height η is assumed to lie on the liner (lower surface in Fig. 1), so that

$$\theta(0, t)\, h(0) = \eta \tag{2.4}$$

because the domain "sees" the lubricant as entering from the left. We assume V to be positive, so that since the right boundary ($x = 1$) is in cavitation, no boundary condition is needed for θ there (information travels towards the right).

Additionally, mass conservation of the lubricant at Σ is to be satisfied. The lubricant flow in the pressurized region is

$$J = -h^3\, \partial_x p + \frac{Sh}{2} \tag{2.5}$$

while that in the cavitated region is

$$J = V\, \theta\, h \tag{2.6}$$

so that the Rankine-Hugoniot conservation conditions read

$$\left\{V\, \theta(\alpha(t)^-, t)\, h(\alpha(t))\right\} - \left\{-h(\alpha(t))^3\, \partial_x p(\alpha(t)^+, t) + \frac{Sh(\alpha(t))}{2}\right\} \;=\; \alpha'(t)\, h(\alpha(t))\, \left\{\theta(\alpha(t)^-, t) - 1\right\} \tag{2.7}$$

$$\left\{-h(\beta(t))^3\, \partial_x p(\beta(t)^-, t) + \frac{Sh(\beta(t))}{2}\right\} - \left\{V\, \theta(\beta(t)^+, t)\, h(\beta(t))\right\} \;=\; \beta'(t)\, h(\beta(t))\, \left\{1 - \theta(\beta(t)^+, t)\right\} \tag{2.8}$$

The previous model essentially amounts to a specific set of cavitation boundary conditions, which are known as JFO conditions and attributed to Jakobson and Floberg [17] and Olsson [21]. If the transport velocity V is taken as $S/2$, the model admits a unique solution in many practical cases, as has been proved and discussed extensively by Bayada and coworkers[5, 6].

Further, the previous model (with $V = S/2$) can be recast into a single equation for θ and p, as proposed by Elrod and Adams[14], by defining $p = 0$ in the cavitated region. This leads to the well-known Elrod-Adams model,

$$\partial_x \left(h^3(x) \partial_x p(x, t) - \frac{S}{2}\, \theta(x, t) h(x) \right) = \partial_t \left[\theta(x, t) h(x)\right] \qquad \text{with } x \in \Omega,\ t \geq 0 \tag{2.9}$$

with the conditions

$$p \geq 0, \qquad 0 \leq \theta \leq 1, \qquad \theta(x, t) < 1 \;\Rightarrow\; p(x, t) = 0, \qquad p(x, t) > 0 \;\Rightarrow\; \theta(x, t) = 1 \tag{2.10}$$

which, understood in a weak sense, automatically incorporates the conservation conditions (2.7)-(2.8).

The Elrod-Adams model is unfortunately not very suitable for the piston-ring problem. Consider the left part of the cavitated region, $0 < x < \alpha(t)$, to begin with, as depicted in Fig. 1. In that region the Elrod-Adams model reduces to the equation

$$\partial_t(\theta\, h) + \frac{S}{2}\, \partial_x(\theta\, h) = 0 \qquad \text{for } 0 < x < \alpha(t) \tag{2.11}$$

for θ. However, the height of the fluid film (i.e., $\theta\, h$) is simply given by amount of fluid already sitting on the cylinder wall and approaching the ring with velocity S. *The transport velocity* V *must thus equal* S *there*. In other words, the field θ must satisfy

$$\partial_t(\theta\, h) + S\, \partial_x(\theta\, h) = 0 \qquad \text{for } 0 < x < \alpha(t) \tag{2.12}$$

and not (2.11).

Consider now the right part of the cavitated region, $\beta(t) < x < 1$. The point $\beta(t)$ is a *rupture boundary*, at which some of the fluid detaches from the wall because otherwise the pressure would become negative. It is not clear whether after the rupture boundary the fluid remains attached to the upper or lower surfaces. Most probably a complex patterns develops locally. In Fig. 1 the fluid has been drawn as always attached to the lower surface, though in principle some of it could attach to the upper one. In any case, further away from $\beta(t)$ a certain film height remains attached to the cylinder wall and travels at speed S towards the right and the Elrod-Adams model again assigns to it a speed $S/2$. Typically, this film height will encounter another ring of the ring pack, so that it is important to compute θ to the right of $\beta(t)$ accurately.

As a consequence of the above, a model allowing for the transport velocity V in the cavitated region $\Omega_0(t)$ to be different from $S/2$ (in particular, equal to S) is crucial for the ring/liner contact problem. In this article, we aim at establishing a well-posed mathematical model with $V = aS$ for $a \neq 1/2$. In particular, we show that, if $1/2 < a \leq 1$, the model consisting of (2.1)-(2.2) together with the interface conditions (2.3) and (2.7)-(2.8) is not well-posed because uniqueness of the solution fails. The need for an additional condition at rupture boundaries is proved, together with the interval within which this condition must be chosen. The proposed model coincides with the Elrod-Adams model when $a = 1/2$ and is a natural extension of it for the range $a \in [1/2, 1]$, therefore much improving the modeling capabilities for ring/liner lubrication problems.

From the viewpoint of a global model, since the mass-conservation conditions (2.7)-(2.8) are to be preserved, we aim at building a well-posed model of the form

$$\partial_x \left(h^3(x)\partial_x p(x,t) - V(\theta)\,\theta(x,t)h(x)\right) = \partial_t\left[\theta(x,t)h(x)\right] \qquad \text{with } x \in \Omega,\ t \geq 0 \tag{2.13}$$

where

$$V(\theta) = \begin{cases} \frac{S}{2} & \text{if } \theta = 1 \\ aS & \text{if } \theta < 1 \end{cases} \tag{2.14}$$

with $a \in \left[\frac{1}{2}, 1\right]$ known. The model is supplemented by conditions (2.10) plus an additional condition that will appear later.

Furthermore, we have the boundary conditions and initials conditions

$$p(0,t) = p(1,t) = 0 \tag{2.15}$$

$$\theta(0,t) = \theta_{\text{in}}(t) \tag{2.16}$$

$$\theta(x,0) = \theta_0(x) \tag{2.17}$$

with θ_0 and θ_{in} known.

2.2. **Establishment of a model in the general case** $a \in \left[\frac{1}{2}, 1\right]$**.** For $a > \frac{1}{2}$, the problem (2.13)-(2.17) is ill-posed. It will become evident in what follows that multiple solutions exist. This was already suggested by Organisciak [22], and later Ausas [2], who numerically solved this problem and obtained highly oscillatory results close to $\beta(t)$, indicative of an ill–posed underlying exact problem.

We assume in this article that $\Omega_+(t)$ is of the form $\Omega_+(t) = \left]\alpha(t), \beta(t)\right[$ with $0 < \alpha(t) < \beta(t) < 1$. We assume $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$ given and satisfying $0 < \alpha_0 < \beta_0 < 1$. In Ω_+, the classic Reynolds equation is thus satisfied for the pressure

$$\partial_x \left(h^3(x)\partial_x p(x,t)\right) = \frac{S}{2}\partial_x h(x) \qquad \text{on } \Omega_+(t) \tag{2.18}$$

with the boundary condition

$$p\big(\alpha(t),t\big) = p\big(\beta(t),t\big) = 0 \tag{2.19}$$

As a consequence we have

$$h^3\partial_x p(x,t) = \frac{S}{2}\Big(h(x) - G\big(\alpha(t),\beta(t)\big)\Big) \qquad \forall x \in \Omega_+(t) \tag{2.20}$$

with

$$G(\alpha,\beta)=\frac{\int_\alpha^\beta \frac{1}{h^2(x)}\mathrm{d}x}{\int_\alpha^\beta \frac{1}{h^3(x)}\mathrm{d}x} \tag{2.21}$$

Remark 2.1. *$\alpha(t)$ and $\beta(t)$ must satisfy $p(x,t)\geq 0$, $\forall x\in\Omega_+(t)=]\alpha(t),\beta(t)[$. Lemma 3.1 gives necessary conditions on the existence of α and β.*

In $\Omega_0(t)$, we have the transport equation for θh :

$$\partial_t(\theta(x,t)h(x))+aS\partial_x(\theta(x,t)h(x))=0 \qquad \forall x\in\Omega_0(t) \tag{2.22}$$

At the cavitation boundary $\Sigma(t)=\{\alpha(t),\beta(t)\}$, the conservation conditions (2.7)-(2.8) can be rewritten as

$$\alpha'(t)h(\alpha(t))\big[1-\theta(\alpha^-(t),t)\big] = -h^3(\alpha(t))\frac{\partial p}{\partial x}(\alpha^+(t),t)+\frac{S}{2}h(\alpha(t))\big[1-2a\theta(\alpha^-(t),t)\big] \tag{2.23}$$

$$\beta'(t)h(\beta(t))\big[1-\theta(\beta^+(t),t)\big] = -h^3(\beta(t))\frac{\partial p}{\partial x}(\beta^-(t),t)+\frac{S}{2}h(\beta(t))\big[1-2a\theta(\beta^+(t),t)\big] \tag{2.24}$$

We assume that the given costant $\eta>0$ satisfies $\eta\in\big]0,\min_{x\in[0,\alpha_0]}h(x)\big[$. To simplify, we suppose that the fluid film thickness to the left of the contact is uniform and equal to η, which means $\theta_0(x)h(x)=\eta$ for all $x\in[0,\alpha_0]$ and $\theta_{\text{in}}(t)h(0)=\eta$ for all $t\geq 0$.

We deduce from (2.22) that $\theta(x,t)=\dfrac{\eta}{h(x)}$ for all $x<\alpha(t)$.
So, we have

$$\theta\big(\alpha^-(t),t\big)=\frac{\eta}{h(\alpha(t))} \tag{2.25}$$

If we replace (2.20) and (2.25) in (2.23), we deduce the differential equation satisfied by $\alpha(t)$, which reads

$$\alpha'(t)=\frac{S}{2}\frac{G\big(\alpha(t),\beta(t)\big)-2a\eta}{h(\alpha(t))-\eta} \tag{2.26}$$

The differential equation satisfied by $\beta(t)$, in turn, depends on $\theta(\beta^+(t),t)$. From (2.24) and (2.20), we have

$$\beta'(t)=\frac{S}{2\big[1-\theta(\beta^+(t),t)\big]}\left(\frac{G\big(\alpha(t),\beta(t)\big)}{h(\beta(t))}-2a\theta(\beta^+(t),t)\right) \qquad \text{if } \theta\big(\beta^+(t),t\big)<1 \tag{2.27}$$

There exists two different situations to express $\theta(\beta^+,t)$, depending on the moving speed of the boundary $x=\beta(t)$:

Case 1 : $\beta'\geq aS$ (this case corresponds to film re-formation, that is, to the growth of the active zone)
In this case, we can find $\theta(\beta^+,t)$ by tracking back the characteristic lines of (2.22) and using the initial value of $\theta(x,t)$. More precisely, the characteristic line through $(\beta(t),t)$ is

$$\tau\mapsto\beta(t)+aS(\tau-t), \qquad \forall\tau\in[0,t[$$

We have two different situations:

(a) $\beta(t)+aS(\tau-t)\geq\beta(\tau)$, $\forall\tau\in[0,t[$ (see fig. 2)
In this case, we have:

$$\theta(\beta^+(t),t)h(\beta^+(t))=\theta(\beta(t)-aSt,0)h(\beta(t)-aSt)$$

leading to

$$\theta\big(\beta^+(t),t\big)=\frac{\theta_0\big(\beta(t)-aSt\big)h\big(\beta(t)-aSt\big)}{h\big(\beta(t)\big)} \tag{2.28}$$

(b) There exists $t_1\in]0,t[$ such that (see fig. 3)

$$\begin{aligned}\beta(t)+aS(t_1-t)&=\beta(t_1)\\ \beta(t)+aS(\tau-t)&>\beta(\tau)\quad\forall\tau\in]t_1,t[\end{aligned}$$

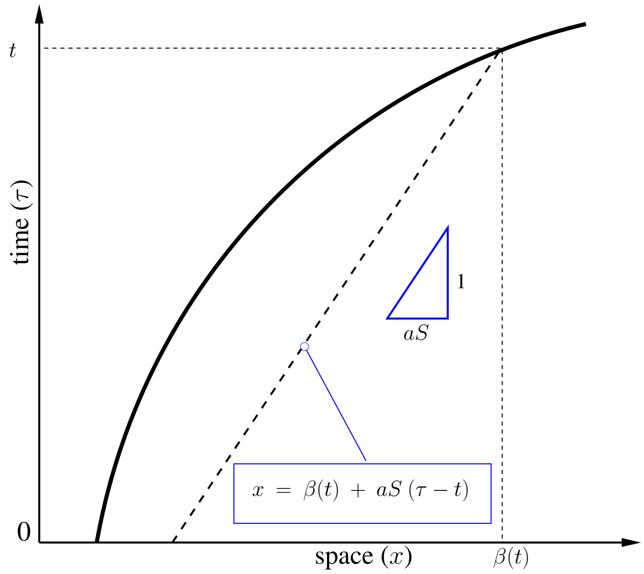

FIGURE 2. Representation of β as a function of time and the characteristic line emanating from $(t, \beta(t))$ as in case 1(a).

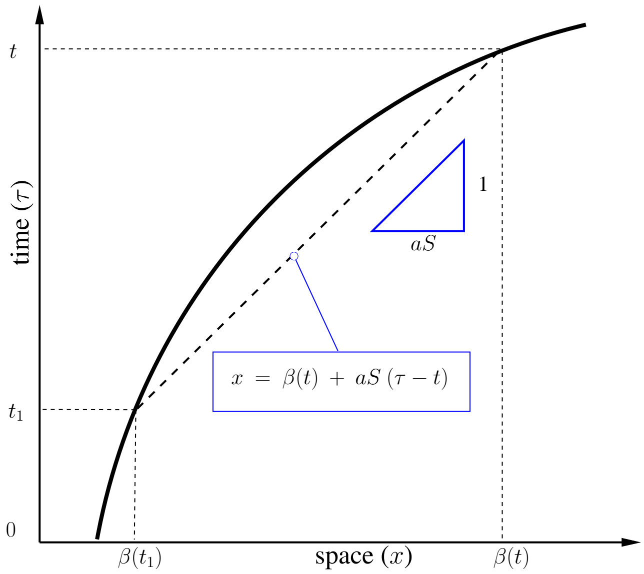

FIGURE 3. Representation of β as a function of time and the characteristic line emanating from $(t, \beta(t))$ as in case 1(b), which intersects the curve at time t_1.

which gives

$$\theta(\beta^+(t), t)h\big(\beta^+(t)\big) = \theta(\beta^+(t_1), t_1)h(\beta(t_1))$$

and thus

$$\theta(\beta^+(t), t) = \frac{\theta\big(\beta^+(t_1), t_1\big)h\big(\beta(t_1)\big)}{h\big(\beta(t)\big)} \tag{2.29}$$

Case 2 : $\beta'(t) < aS$ (this case corresponds to film rupture)
Since $\partial_x p(\beta^-(t), t) \leq 0$ (because $p \geq 0$ on $\Omega_+(t)$), we obtain from (2.24) that

$$\theta(\beta^+(t), t) \geq \frac{\frac{S}{2} - \beta'(t)}{aS - \beta'(t)} = F(\beta'(t)) \tag{2.30}$$

with $F :]-\infty, aS[\longrightarrow \mathbb{R}$ defined by $F(z) = \dfrac{\frac{S}{2} - z}{aS - z}$.

Remark 2.2. *Condition (2.30) implies that $\theta(\beta^+(t), t)$ must satisfy*

$$\max\{0, F(\beta'(t))\} \leq \theta(\beta^+(t), t) \leq 1 \tag{2.31}$$

If $a = \frac{1}{2}$, we have $F(z) = 1$ for all z and thus this condition reduces to the equation $\theta(\beta^+, t) = 1$. If $a > \frac{1}{2}$ then $\theta(\beta^+(t), t)$ is not fully determined by condition (2.31). In fact, the model as it is only states that $\theta(\beta^+(t), t)$ must belong to the shaded area in Figure 4. For any given β' there is a finite interval of values of $\theta(\beta^+(t), t)$ that satisfies (2.31), and each of these values leads to a different solution of the problem. This explains the lack of uniqueness of the model presented so far when $a \neq \frac{1}{2}$, and the need for an additional condition to have a unique solution.

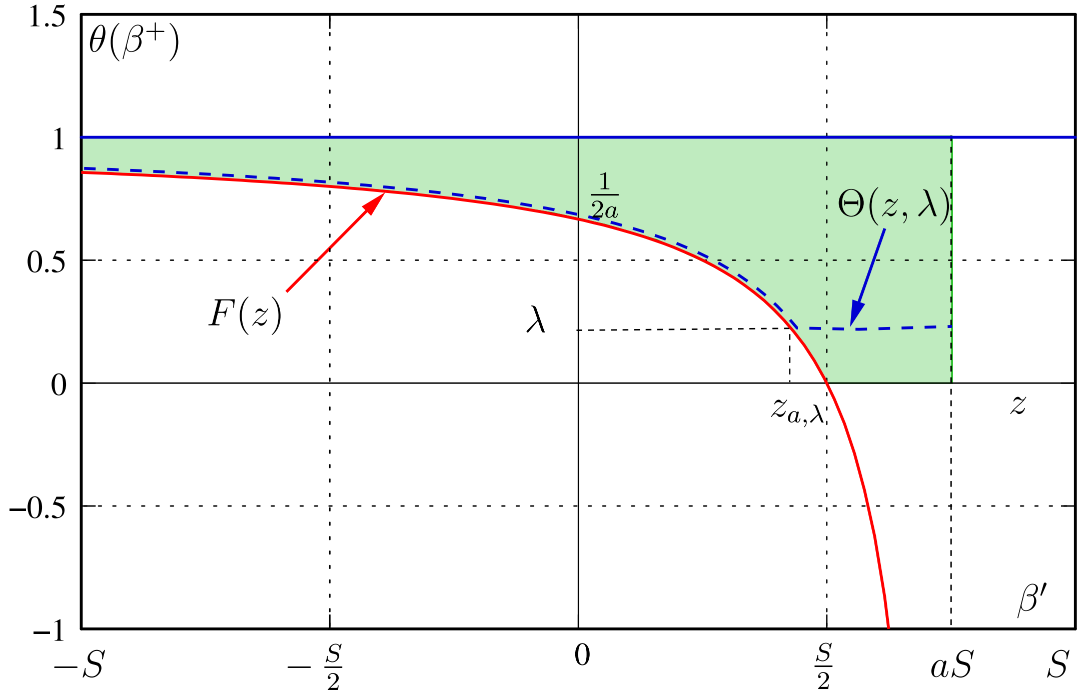

FIGURE 4. Graphical representation (in red) of $F(z)$ for $a \in]0.5; 1]$, in particular the value $a = 0.75$ has been taken. The green shaded area corresponds to acceptable values of $(\beta'(t), \theta(\beta^+, t))$. Also shown are, for some specified λ, the velocity $z_{a,\lambda}$ (such that $F(z_{a,\lambda}) = \lambda$) and the function $\Theta(z, \lambda) = \max\{\lambda, F(z)\}$.

Recovering uniqueness: Among the multiple values possible for $\theta(\beta^+(t), t)$ in the case $\beta'(t) < aS$ (rupture), we propose here a specific choice that leads to a well-posed problem. The basic idea is to choose $\theta(\beta^+(t), t)$ equal to some given value λ, unless $F(\beta'(t)) > \lambda$, in which case the minimum allowed value (i.e.; $F(\beta'(t))$) is taken.

Remark 2.3. *We believe that more physically realistic models for the rupture lubricant content $\theta(\beta^+(t), t)$ will be developed in the future, on the basis of experiments and detailed multiphase Navier-Stokes simulations. In fact, it could well happen that this variable is history-dependent and obeys a differential equation of its own. Our contribution here is restricted to unveiling the origin of the lack of uniqueness when $a \neq \frac{1}{2}$, to proposing a simple fix of this* *difficulty and to proving the well-posedness of the resulting model. It should be noted that the proposed choice for* $\theta(\beta^+(t),t)$*, if* $\lambda \leq \frac{1}{2a}$*, yields the* same steady state pressure field *as the Elrod-Adams model, when the models are run with the same lubricant flow* J.

Let $\lambda \in \big[0,1\big]$ be a parameter, and let us define

$$\Theta(z,\lambda) = \max\{\lambda, F(z)\} \tag{2.32}$$

We choose $\theta\big(\beta^+(t),t\big)$ according to

$$\theta\big(\beta^+(t),t\big) = \Theta(\beta'(t),\lambda) = \max\big\{F\big(\beta'(t)\big),\lambda\big\} \tag{2.33}$$

i.e.;

$$\theta(\beta^+(t),t) = \begin{cases} \frac{\frac{S}{2}-\beta'(t)}{aS-\beta'(t)} & \text{if } \beta'(t) \leq z_{a,\lambda} \\ \lambda & \text{if } \beta'(t) > z_{a,\lambda} \end{cases} \tag{2.34}$$

with

$$z_{a,\lambda} = \begin{cases} S\frac{a\lambda-\frac{1}{2}}{\lambda-1} & \text{if } \lambda \neq 1 \\ -\infty & \text{if } \lambda = 1 \text{ and } a \neq \frac{1}{2} \\ \frac{S}{2} & \text{if } \lambda = 1 \text{ and } a = \frac{1}{2} \end{cases} \tag{2.35}$$

See also Fig. 4 for graphical interpretation. From (2.24) and (2.34), we obtain

(i) If $\beta'(t) \leq z_{a,\lambda}$ then, from (2.24), we deduce $\frac{\partial p}{\partial x}(\beta^-(t),t) = 0$. Then, according to (2.20), this implies that $\alpha(t)$ and $\beta(t)$ satisfy the algebraic equation

$$G(\alpha(t),\beta(t)) = h(\beta(t)) \tag{2.36}$$

(ii) If $z_{a,\lambda} < \beta'(t) < aS$ (this case doesn't exist if $a = \frac{1}{2}$), we deduce from (2.27) and (2.34) that

$$\beta'(t) = \frac{S}{2(1-\lambda)}\left(\frac{G(\alpha(t),\beta(t))}{h\big(\beta(t)\big)} - 2a\lambda\right) \tag{2.37}$$

Collecting the previous results, the model proposed here can be summarized as determining (p,θ) that (weakly) satisfy

$$\partial_x\left(h^3(x)\partial_x p(x,t) - V(\theta)\,\theta(x,t)h(x)\right) = \partial_t\left[\theta(x,t)h(x)\right] \qquad \text{with } x\in\Omega,\ t\geq 0 \tag{2.38}$$

with

$$p \geq 0, \qquad 0 \leq \theta \leq 1, \qquad \theta(x,t) < 1 \Rightarrow p(x,t) = 0, \qquad p(x,t) > 0 \Rightarrow \theta(x,t) = 1, \tag{2.39}$$

$$V(\theta) = \begin{cases} \frac{S}{2} & \text{if } \theta = 1 \\ aS & \text{if } \theta < 1 \end{cases} \tag{2.40}$$

and

$$\theta = \Theta(\beta',\lambda) \qquad \text{at rupture boundaries} \tag{2.41}$$

where β' denotes the (unknown) velocity of the rupture boundary, and $a \in \big[\frac{1}{2},1\big]$, $\lambda \in [0,1]$ are user-defined model parameters. It is understood that suitable initial and boundary conditions are imposed, which in fact can be chosen such that there are two interfaces, $\alpha(t)$ and $\beta(t)$. In this case the previous model is equivalent to the following algebraic-differential system for $(\alpha(t),\beta(t))$:

$$\alpha'(t) = \frac{S}{2}\frac{G\big(\alpha(t),\beta(t)\big) - 2a\eta}{h\big(\alpha(t)\big) - \eta} \tag{2.42}$$

with

$$\text{If } \beta'(t) \leq z_{a,\lambda} \qquad \text{then } G\big(\alpha(t),\beta(t)\big) = h\big(\beta(t)\big) \tag{2.43}$$

$$\text{If } z_{a,\lambda} < \beta'(t) \leq aS \qquad \text{then } \beta'(t) = \frac{S}{2(1-\lambda)}\left(\frac{G\big(\alpha(t),\beta(t)\big)}{h\big(\beta(t)\big)} - 2a\lambda\right) \tag{2.44}$$

$$\text{If } aS < \beta'(t) \qquad \text{then } \beta'(t) = \frac{S}{2\big(1-\theta(\beta^+(t),t)\big)}\left(\frac{G\big(\alpha(t),\beta(t)\big)}{h\big(\beta(t)\big)} - 2a\theta(\beta^+(t),t)\right) \tag{2.45}$$

where $\theta(\beta^+(t),t)$ is calculated tracing back the characteristic line, as shown before (**case** 1).

with the condition that $\big(\alpha(t),\beta(t)\big)$ must be such that $p(x,t) \geq 0\ \forall x \in [\alpha(t),\beta(t)]$.

Remark 2.4. *Even if all the data are time independent, the previous model (and in particular the Elrod–Adams model) yields solutions $p(x,t)$ and $\theta(x,t)$ that in general depend on time. The initial saturation function $\theta(x,0)$ does not in general make the left-hand sides of (2.7) and (2.8) to vanish, so that the cavitation boundary evolves with time and, as a consequence, so do the solution functions p and θ.*

3. Mathematical study of the problem

We add the following hypotheses:

(H1) $\exists \gamma \in]0,1[$ such as :
- h is strictly decreasing on $]0,\gamma[$
- h is strictly increasing on $]\gamma,1[$
- $\min_{x\in[0,1]} h(x) = h(\gamma) = h_{\min} > 0$
- h is continuous on $[0,1]$.

(H2) The constants variables α_0, β_0 satisfy :
$0 < \alpha_0 < \gamma < \beta_0 < 1$

The goal of this section is to give a complete mathematical study of problem (2.42)-(2.45). In section 3.1 we define the admissible domain $A_1 \subset \mathbb{R}^2$ of $(\alpha(t),\beta(t))$. On the other hand Lemmata 3.1, 3.2, 3.3 and 3.4 give a complete characterization of A_1. We also characterize in lemma 3.5 and lemma 3.6 the domain $A_2 \subseteq A_1$ where we have $z_{a,\lambda} < \beta'(t) < aS$, that is, the ODE (2.44) must be used. In section 3.2 the existence of a stationary solution is studied, while in section 3.3 we study the existence of a global solution in time using the hypothesis (3.18) which assumes that $A_1 = A_2$. This means tha the evolution of the system is given only by two ODEs, (2.42) and (2.44), or one ODE (2.42) and one algebraic equation (2.43).

3.1. Definition of the admissible region .

In the previous sections the condition $p \geq 0$ was mentioned several times as a *restriction* on the possible values of $\alpha(t)$ and $\beta(t)$. This restriction is in fact parametric in time in the sense that, for some given instant t, the instantaneous values $\alpha(t)$ and $\beta(t)$ either lead to $p \geq 0$ or not, irrespective of their history. Here we work out the details needed to make this restriction on $\alpha(t)$ and $\beta(t)$ explicit. For that purpose, α and β are treated as independent variables in what follows, satisfying $0 < \alpha < \gamma < \beta < 1$. Considering thus α and β given, we study the solutions $p = p_{\alpha,\beta}$ of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3(x)\frac{dp}{dx}(x)\right) = \frac{S}{2}\frac{\mathrm{d}h(x)}{\mathrm{d}x} \qquad x \in]\alpha,\beta[\tag{3.1}$$

$$p(\alpha) = p(\beta) = 0 \tag{3.2}$$

which provide us with the pressure field that would develop in the pressurized region if, at some instant t, its left boundary is located at $x = \alpha$ and its right boundary at $x = \beta$. Knowing $p_{\alpha,\beta}$ then allows us to analyze possible solutions to (2.42)-(2.45), which are nothing but orbits $(\alpha(t),\beta(t))$ in the $\alpha - \beta$ plane.

Lemma 3.1. *We have $p(x) \geq 0$ for all $x \in [\alpha,\beta]$ if and only if*

$$h(\beta) \leq G(\alpha,\beta) \leq h(\alpha) \tag{3.3}$$

Proof. From (3.1) we have, as in (2.20),

$$h^3(x)p'(x) = \frac{S}{2}\Big(h(x) - G\big(\alpha,\beta\big)\Big) \tag{3.4}$$

As a consequence,

- if $p(x) \geq 0$, for any $x \in [\alpha,\beta]$ then $p'(\alpha) \geq 0$ and $p'(\beta) \leq 0$, which gives (3.3).
- If (3.3) is satisfied then, from hypothesis (H1) the function $x \mapsto g(x) := h(x) - G\big(\alpha,\beta\big)$ vanishes at two points, $x = x_1$ and $x = x_2$, and is negative in the interval $]x_1, x_2[$. Assuming that (3.3) holds, then $0 \leq \alpha \leq x_1$ and $x_1 \leq \beta \leq x_2$. As a consequence, $p'(x) \geq 0$ for $x \in [\alpha, x_1]$, and $p'(x) \leq 0$ for $x \in [x_1,\beta]$. This implies that $p(x) \geq 0$ for $x \in [\alpha,\beta]$, with a maximum at x_1.

□

Let us define

$$\begin{aligned} A_1 &= \{\ (\alpha,\beta) \mid 0 < \alpha < \gamma < \beta < 1 \text{ and } p_{\alpha,\beta}(x) \geq 0, \forall x \in [\alpha,\beta]\ \} \\ &= \{\ (\alpha,\beta) \mid 0 < \alpha < \gamma < \beta < 1 \text{ and } h(\beta) \leq G(\alpha,\beta) \leq h(\alpha)\ \} \end{aligned} \tag{3.5}$$

and compute the derivatives of G,

$$\frac{\partial G}{\partial \alpha}(\alpha,\beta) = \frac{G(\alpha,\beta) - h(\alpha)}{h^3(\alpha)\int_\alpha^\beta \frac{\mathrm{d}\xi}{h^3(\xi)}} \qquad \frac{\partial G}{\partial \beta}(\alpha,\beta) = \frac{h(\beta) - G(\alpha,\beta)}{h^3(\beta)\int_\alpha^\beta \frac{\mathrm{d}\xi}{h^3(\xi)}} \tag{3.6}$$

which show that G is a strictly decreasing function of both α and β inside of A_1. Notice also that $G(\alpha,\beta)$ can be extended by continuity to (γ,γ) as

$$G(\gamma,\gamma) = h(\gamma) \tag{3.7}$$

Now consider the following function:

$$\begin{array}{rccc} f: & [0,\gamma]\times[\gamma,1] & \to & \mathbb{R} \\ & (\alpha,\beta) & \mapsto & \displaystyle\int_\alpha^\beta \frac{\mathrm{d}\xi}{h^2(\xi)} - h(\beta)\int_\alpha^\beta \frac{\mathrm{d}\xi}{h^3(\xi)} \end{array} \tag{3.8}$$

Lemma 3.2. *If* $f(0,1) \leq 0$ *then* $f(\alpha,1) < 0$ *for any* $\alpha \in]0,\gamma]$.
If $f(0,1) > 0$ *then there exists* $\overline{\alpha} \in \big]0,\gamma\big[$ *such that* $f(\alpha,1) = \begin{cases} > 0 & \text{if } \alpha < \bar\alpha \\ = 0 & \text{if } \alpha = \bar\alpha \\ < 0 & \text{if } \alpha > \bar\alpha \end{cases}$

Proof. Let $f_1 : [0,\gamma] \to \mathbb{R}$ be given by $f_1(\alpha) = f(\alpha,1)$. We have

$$f_1'(\alpha) = -\frac{1}{h^2(\alpha)} + \frac{h(1)}{h^3(\alpha)} = \frac{1}{h^3(\alpha)}\left[h(1) - h(\alpha)\right]$$

and thus

$$f_1(\gamma) = \int_\gamma^1 \frac{h(x)}{h^3(x)}\mathrm{d}x - h(1)\int_\gamma^1 \frac{1}{h^3(x)}\mathrm{d}x < 0$$

because h is strictly increasing in $]\gamma,1]$. We have two different cases :

Case 1 : $h(1) \geq h(0)$, then $f_1'(\alpha) \geq 0$, for any $\alpha \in]0,\gamma]$ because $h(\alpha) < h(0)$. In this case, $f_1(\gamma) < 0$ proves that $f_1(\alpha) < 0$ for all $\alpha \in [0,\gamma]$.

Case 2 : $h(1) < h(0)$, then there exist $\hat\alpha \in]0,\gamma[$ such that $h(\hat\alpha) = h(1)$ and then f_1 is strictly decreasing on $[0,\hat\alpha]$ and strictly increasing on $[\hat\alpha,\gamma]$. This gives the result.

□

The following lemma provides a characterization of A_1.

Lemma 3.3.

1) If $G(0,1) \le h(1)$ then for any $\alpha \in]0,\gamma[$ there exists a unique solution $\beta^* = \beta^*(\alpha) \in]\gamma,1[$ of the equation

$$G(\alpha,\beta^*) = h(\beta^*) \tag{3.9}$$

In this case,

$$A_1 = \{ (\alpha,\beta) \mid 0 < \alpha < \gamma \text{ and } \gamma < \beta \le \beta^*(\alpha) \} \tag{3.10}$$

2) If $G(0,1) > h(1)$ then there exists $\bar\alpha \in]0,\gamma[$ such that for all $\alpha \in [\bar\alpha,\gamma[$ there exists a unique solution $\beta^* = \beta^*(\alpha) \in]\gamma,1]$ of (3.9). In this case :

$$A_1 = \{ (\alpha,\beta) \mid \bar\alpha \le \alpha < \gamma \text{ and } \gamma < \beta \le \beta^*(\alpha) \} \cup]0,\bar\alpha] \times]\gamma,1[\tag{3.11}$$

In both cases we have for all α

$$h\big(\beta^*(\alpha)\big) < h(\alpha) \tag{3.12}$$

Proof. Let $\alpha \in [0,\gamma[$ be given. Finding $\beta \in]\gamma,1]$ such that $G(\alpha,\beta) \ge h(\beta)$ is as finding $\beta \in]\gamma,1]$ such that $f(\alpha,\beta) \ge 0$ with f given by (3.8). We have

$$\frac{\partial f}{\partial \beta}(\alpha,\beta) = -h'(\beta)\int_\alpha^\beta \frac{d\xi}{h^3(\xi)} < 0, \qquad f(\alpha,\gamma) = \int_\alpha^\gamma \frac{1}{h^3(\xi)} h(\xi)d\xi - h(\gamma)\int_\alpha^\gamma \frac{1}{h^3(\xi)}d\xi > 0$$

because $h(x) > h(\gamma)$ (hypothesis (H1)). If $f(0,1) \le 0$, that is $G(0,1) \le h(1)$, then by lemma 3.2 $f(\alpha,1) < 0$, for any $\alpha < \gamma$ and then there exists a unique $\beta^* = \beta^*(\alpha)$ solution of (3.9) and we also have

$$G(\alpha,\beta) \ge h(\beta) \iff \beta < \beta^*(\alpha)$$

If $f(0,1) > 0$, that is $G(0,1) > h(1)$, we deduce from lemma 3.2 that

$$\alpha < \bar\alpha \implies G(\alpha,\beta) > h(\beta)\ \forall \beta \in [\gamma,1]$$

If $\alpha \ge \bar\alpha$, we have the same results as in the case $f(0,1) \le 0$.

We now prove (3.12). Suppose by contradiction that $h(\beta^*) \ge h(\alpha)$. This entails

$$f(\alpha,\beta^*(\alpha)) = \int_\alpha^{\beta^*} \frac{h(\xi)}{h^3(\xi)}d\xi - h(\beta^*)\int_\alpha^{\beta^*} \frac{1}{h^3(\xi)}d\xi < 0$$

because $h(x) < h(\beta^*)$ for all $x \in]\alpha,\beta^*[$. This contradicts the definition of $\beta^*(\alpha)$, which states $f\big(\alpha,\beta^*(\alpha)\big) = 0$. It remains to prove that, if A_1 is given by (3.10-3.11), then $G(\alpha,\beta) \le h(\alpha)$. But if $(\alpha,\beta) \in A_1$ then $h(\beta) \le G(\alpha,\beta)$ and from (3.5) we have $\dfrac{\partial G}{\partial \beta} \le 0$. It is thus sufficient to prove that $G(\alpha,\gamma) \le h(\alpha)$; i.e., that $\displaystyle\int_\alpha^\gamma \frac{1}{h^2(\xi)}d\xi \le h(\alpha)\int_\alpha^\gamma \frac{1}{h^3(\xi)}d\xi$. Since $\dfrac{1}{h^2(x)} \le \dfrac{h(\alpha)}{h^3(x)}$ for all $x \in [\alpha,\gamma[$, this inequality is verified. □

We have the following result:

Lemma 3.4. *Let β^* the function defined in lemma 3.3 with domain $]0,\gamma[$ for $G(0,1) \le h(1)$ or $[\bar\alpha,\gamma[$ for $G(0,1) > h(1)$. Then β^* has the following properties :*

i) $\beta^* \in C^1$
ii) $(\beta^*(\alpha))' < 0$
iii) $\lim_{\substack{\alpha\to\gamma \\ \alpha<\gamma}} \beta^*(\alpha) = \gamma$

Proof. By definition, $\beta^*(\alpha)$ satisfies $f(\alpha,\beta^*(\alpha)) = 0$, with f defined by (3.8). We have that

$$\frac{\partial f}{\partial \beta}(\alpha,\beta^*(\alpha)) = -h'(\beta^*(\alpha))\int_\alpha^{\beta^*} \frac{d\xi}{h^3(\xi)} < 0$$

Since $f \in C^1$, the implicit function theorem implies that $\beta^* \in C^1$, proving $i)$.
Also,

$$\frac{\partial f}{\partial \alpha}(\alpha,\beta^*(\alpha)) = \frac{h(\beta^*(\alpha)) - h(\alpha)}{h^3(\alpha)} < 0 \qquad \text{from (3.12)}$$

which, together with $\frac{\partial f}{\partial \alpha}(\alpha, \beta^*(\alpha)) + (\beta^*(\alpha))' \frac{\partial f}{\partial \beta}(\alpha, \beta^*(\alpha)) = 0$, implies $ii)$.
The assertion $iii)$ is a consequence of (3.12). □

Let us now reformulate the condition $z_{\alpha,\lambda} \leq \beta' \leq aS$ in the case $a > \frac{1}{2}$. We can easily deduce from (2.44) that

$$z_{a,\lambda} \leq \beta' \leq aS \iff h(\beta) \leq G(\alpha, \beta) \leq 2ah(\beta) \tag{3.13}$$

Therefore we define

$$A_2 = \{ \ (\alpha, \beta) \in A_1 \mid 0 < \alpha < \gamma < \beta < 1 \text{ and } G(\alpha, \beta) \leq 2ah(\beta) \ \} \tag{3.14}$$

From now on, we assume to avoid technicalities that h is such that

$$G(0, 1) \leq h(1) \tag{3.15}$$

so that we are in the setting of the lemma 3.3 1). In particular there exists a unique $\beta^*(\alpha)$ and

$$A_1 = \{ \ (\alpha, \beta) \mid 0 < \alpha < \gamma, \gamma < \beta \leq \beta^*(\alpha) \ \}$$

Assumption (3.15) is true, for example, if $h(0) \leq h(1)$, because $G(0, 1) = \dfrac{\int_0^1 \frac{h(\xi)}{h^3(\xi)} d\xi}{\int_0^1 \frac{1}{h^3(\xi)} d\xi} < h(1)$.

Lemma 3.5. *Suppose that $a \in \,]\frac{1}{2}, 1]$.*

a) If $G(0, \gamma) \leq 2ah(\gamma)$ then $A_1 = A_2$.
b) If $G(0, \gamma) > 2ah(\gamma)$ then there exist $\underline{\alpha} \in \,]0, \gamma[$ such that $\forall \alpha \in \,]0, \underline{\alpha}[$, $\exists \beta_* = \beta_*(\alpha) \in \,]\gamma, \beta^*(\alpha)[$ solution of the equation $G(\alpha, \beta_*) = 2ah(\beta_*)$. In this case, we have :

$$A_2 = \{ \ (\alpha, \beta) \mid 0 < \alpha \leq \underline{\alpha}, \beta_*(\alpha) \leq \beta \leq \beta^*(\alpha) \ \} \cup \{ \ (\alpha, \beta) \mid \underline{\alpha} < \alpha < \gamma, \gamma < \beta \leq \beta^*(\alpha) \ \}$$

Proof. Let $R(\alpha, \beta) := G(\alpha, \beta) - 2ah(\beta)$, defined in A_1. From (3.6) and (H1), R is strictly decreasing in α with β given and constant, and also strictly decreasing in β with α given and constant. We thus have that $R(\alpha, \beta) \leq R(0, \gamma) = G(0, \gamma) - 2ah(\gamma)$ for all $(\alpha, \beta) \in A_1$. If $R(0, \gamma) \leq 0$ this proves item (a). If $R(0, \gamma) > 0$, since from (3.7) we have that $R(\gamma, \gamma) = (1 - 2a)h(\gamma) < 0$, there exists $\underline{\alpha}$ such that $R(\alpha, \gamma) = \begin{cases} > 0 & \text{if } \alpha < \underline{\alpha} \\ = 0 & \text{if } \alpha = \underline{\alpha} \\ < 0 & \text{if } \alpha > \underline{\alpha} \end{cases}$ and item (b) follows. □

Lemma 3.6. *If $G(0, \gamma) \geq 2ah(\gamma)$ then the function β_* from lemma 3.5 verifies*

i) $\beta_* \in C^1$
ii) $\beta'_* < 0$

This result is an immediate consequence of the function R (see proof of Lemma 3.5) being strictly decreasing in both α and β on the set A_1.

Remark 3.7. *i) If $h(0) \leq 2ah(\gamma)$ then $G(0, \gamma) < 2ah(\gamma)$ (because $G(0, \gamma) < h(0)$).*
ii) If $G(0, \gamma) > 2ah(\gamma)$, then $\beta_*(\alpha)$ exists and is unique.

3.2. **Stationary solutions .**

A stationary solution of the model (2.42)-(2.45) is a pair $(\alpha_e, \beta_e) \in [0, \gamma] \times [\gamma, 1]$ which leads to $\alpha'(t) = \beta'(t) = 0$. By algebraic manipulation of (2.42) and (2.43), it follows that α_e and β_e must satisfy

$$G(\alpha_e, \beta_e) = 2a\eta \tag{3.16}$$

$$G(\alpha_e, \beta_e) = h(\beta_e) \tag{3.17}$$

We have the following result :

Proposition 3.8. *Assume hypotheses (H1), (H2), (3.15) and suppose that $h(\gamma) < 2a\eta < h(\beta^*(0))$ then there exists a unique solution (α_e, β_e) of (3.16, 3.17).*

Proof. Since h is continuous and strictly increasing in $]\gamma,1[$ (by hypothesis (H1)) and $2a\eta\ \in]h(\gamma),h(\beta^*(0))[\subset]h(\gamma),h(1)[$, there exists a unique $\beta_e\in]\gamma,1[$ such that $h(\beta_e)=2a\eta$. The existence and uniqueness of $\alpha_e\in]0,\gamma[$ such that (3.17) is verified is equivalent to finding α_e such that $\alpha_e=(\beta^*)^{-1}(\beta_e)$ with $(\beta^*)^{-1}$ the inverse function of β^* defined in Lemma 3.3. From hypothesis (3.15), β^* is defined on $[0,\gamma[$. On the other hand, β^* is strictly decreasing and $\lim\limits_{\substack{\alpha\to\gamma\\ \alpha<\gamma}}\beta^*(\alpha)=\gamma$ from Lemma 3.4. Thus, $(\beta^*)^{-1}$ exists and is defined on $]\gamma,\beta^*(0)[$. The result follows because $\beta_e\in]\gamma,\beta^*(0)[$. □

3.3. **Global solutions in time .** In this section, and to avoid technicalities, we add to assumption (3.15) the assumption

$$G(0,\gamma)<2ah(\gamma) \tag{3.18}$$

which gives, from Lemma 3.5,

$$A_2=A_1=\{\ (\alpha,\beta)\mid 0\le\alpha\le\gamma,\gamma\le\beta\le\beta^*(\alpha)\ \} \tag{3.19}$$

Remark 3.9. *i) Notice that (3.18) can only hold if* $a>\frac{1}{2}$ *(because* $(0,\gamma)\in A_1$ *and thus, from (3.5),* $G(0,\gamma)\ge h(\gamma)$*). The results in this section thus do not apply to the Elrod-Adams model.*
ii) The two hypotheses (3.15) and (3.18) depend just on the function h.

We remind the hypothesis done on η,

$$0<\eta<h(0) \tag{3.20}$$

which means that *the lubricant film at the inflow boundary is incomplete*. This holds in most applications. It certainly holds in the case of piston rings.

3.3.1. *Preliminaries.* We introduce the number $\widetilde{\alpha}\in\big]0,\gamma\big]$ defined by

$$\begin{cases}\widetilde{\alpha}=\gamma & \text{if } h(\gamma)\ge\eta\\ \widetilde{\alpha}=(h|_{[0,\gamma]})^{-1}(\eta) & \text{if } h(\gamma)<\eta<h(0)\end{cases} \tag{3.21}$$

and we define the set

$$A=\{\ (\alpha,\beta)\in A_2\mid\alpha<\widetilde{\alpha}\ \} \tag{3.22}$$

which is depicted in Figure 5. It is clear that, if $(\alpha,\beta)\in A$, then $h(\alpha)>\eta$. From now on, we assume that the initial condition (α_0,β_0) is in A and we prove that $\big(\alpha(t),\beta(t)\big)\in A$. For this purpose, let us decompose the boundary of the set A as

$$\partial A=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4 \text{ with }\begin{cases}\Gamma_1=\{0\}\times[\gamma,\beta^*(0)]\\ \Gamma_2=[0,\widetilde{\alpha}]\times\{\gamma\}\\ \Gamma_3=\{\widetilde{\alpha}\}\times[\gamma,\beta^*(\widetilde{\alpha})]\\ \Gamma_4=\{\ \big(\alpha,\beta^*(\alpha)\big)\mid\alpha\in\big]0,\widetilde{\alpha}\big[\ \}\end{cases}$$

Remark 3.10. *If* $\widetilde{\alpha}=\gamma$ *then* Γ_3 *reduces to a single point* $\big\{(\gamma,\gamma)\big\}$.

In order to study the evolution of $(\alpha(t),\beta(t))$ we define Z as the isoline of G that solves (3.16), i.e.,

$$Z=\{\ (\alpha,\beta)\in A\mid G(\alpha,\beta)=2a\eta\ \}$$

Combining the equality $G\big(0,\beta^*(0)\big)=h\big(\beta^*(0)\big)$ with the fact that G is strictly decreasing in β we have that

$$G(0,\gamma)>h\big(\beta^*(0)\big)$$

and also that

$$2ah(0)>h(0)>G(0,\gamma).$$

The following result characterizes the existence of the isoline Z as a function of the η value:

Lemma 3.11.
a) If $2ah(0)>2a\eta\ge G(0,\gamma)$ *then* $Z=\emptyset$*. In this case,*

$$G(\alpha,\beta)-2a\eta<0\qquad\qquad\forall(\alpha,\beta)\in A$$

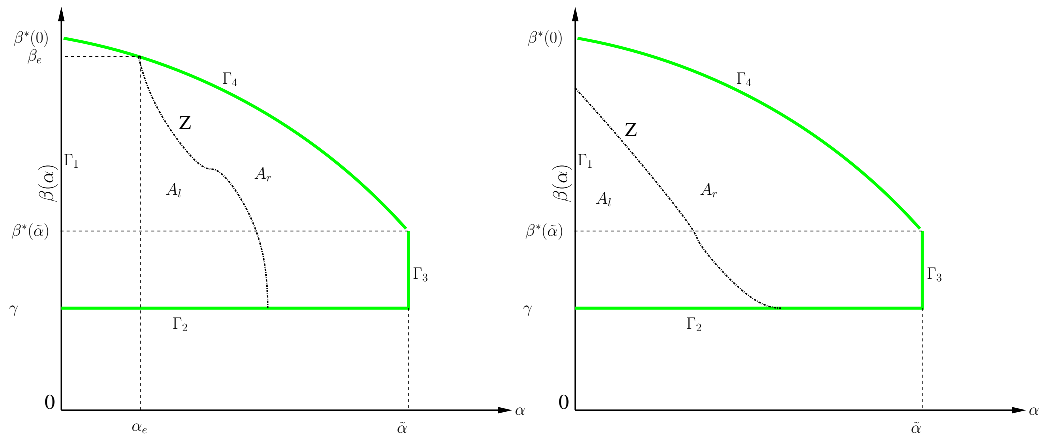

FIGURE 5. Domains A, A_l, A_r and the isoline Z. The case $h(\gamma) < 2a\eta < h(\beta^*(0))$ with stationary solution (α_e, β_e) (left) and the case $h(\beta^*(0)) \leq 2a\eta < G(0, \gamma)$ (right).

b) If $2a\eta \leq h(\gamma)$ then $Z = \emptyset$. In this case,

$$G(\alpha, \beta) - 2a\eta > 0 \qquad \forall(\alpha, \beta) \in A$$

c) If $h(\gamma) < 2a\eta < G(0, \gamma)$, let us define $\alpha_e^0 \in \big[0, \widetilde{\alpha}\big[$ such that

$$\alpha_e^0 = \begin{cases} 0 & \textit{if } G(0,\gamma) > 2a\eta \geq h\big(\beta^*(0)\big) \quad \textit{Case with no stationary solution} \\ \alpha_e & \textit{if } h(\gamma) < 2a\eta < h\big(\beta^*(0)\big) \quad \textit{Case with stationary solution} \end{cases}$$

where α_e is defined in Proposition 3.8. Then there exists $\alpha_I^0 \in \big]\alpha_e^0, \widetilde{\alpha}\big[$ and there exists a function β_I : $\big[\alpha_e^0, \alpha_I^0\big] \to \mathbb{R}$ of class $\mathscr{C}^1$, strictly positive and decreasing, with $\beta_I(\alpha_I^0) = \gamma$, such that the isoline Z exists and is defined by

$$Z = \big\{ \big(\alpha, \beta_I(\alpha)\big) \mid \alpha \in \big[\alpha_e^0, \alpha_I^0\big] \big\}$$

Moreover if $h\big(\beta^*(0)\big) > 2a\eta > h(\gamma)$ we have that $\beta_I(\alpha_e) = \beta_e$. On the other hand, the isoline Z separates A into two parts :

- $G(\alpha, \beta) - 2a\eta > 0 \iff (\alpha, \beta) \in A_\ell$*, A_ℓ :the "left" part of Z*
- $G(\alpha, \beta) - 2a\eta = 0 \iff (\alpha, \beta) \in Z$
- $G(\alpha, \beta) - 2a\eta < 0 \iff (\alpha, \beta) \in A_r$*, A_r :the "right" part of Z*

Proof. We remind that G is strictly decreasing in α and in β on the set A.

Case a) Since $2a\eta \geq G(0, \gamma)$ and since the function $G_1 : \alpha \mapsto G(\alpha, \gamma)$ is strictly decreasing, we have that

$$G(\alpha, \gamma) < 2a\eta \qquad \text{if } \alpha \in \big]0, \widetilde{\alpha}\big[$$

and the result follows.

Case b) If $2a\eta \leq h(\gamma)$ then $h(\gamma) > \eta$ which implies $\widetilde{\alpha} = \gamma$. Now notice that

$$G\big(\alpha, \beta^*(\alpha)\big) - 2a\eta = h\big(\beta^*(\alpha)\big) - 2a\eta > h(\gamma) - 2a\eta \geq 0 \text{ if } \alpha < \gamma$$

and the result follows.

Case c) Assume :

$$G(0, \gamma) > 2a\eta \geq h(\gamma)$$

We remark that for any $\widetilde{\alpha} \leq \gamma$ we have that

$$G(\widetilde{\alpha}, \gamma) = \frac{\displaystyle\int_{\widetilde{\alpha}}^{\gamma} \frac{h(x)}{h^3(x)} dx}{\displaystyle\int_{\widetilde{\alpha}}^{\gamma} \frac{1}{h^3(x)} dx} \leq h(\widetilde{\alpha}) < 2a\eta \tag{3.23}$$

c.1) Assume first that $G(0,\gamma) > 2a\eta \geq h\big(\beta^*(0)\big)$. We deduce from (3.23) that there exists $\alpha_I^0 \in \,]0,\widetilde{\alpha}[$ such that

$$G(\alpha,\gamma) = \begin{cases} > 2a\eta & \text{if } \alpha < \alpha_I^0 \\ < 2a\eta & \text{if } \alpha > \alpha_I^0 \\ 2a\eta & \text{if } \alpha = \alpha_I^0 \end{cases}$$

Also notice that

$$G\big(\alpha,\beta^*(\alpha)\big) - 2a\eta = h\big(\beta^*(\alpha)\big) - 2a\eta < h\big(\beta^*(0)\big) - 2a\eta \leq 0,$$

because the function $\beta^*(\alpha)$ is decreasing on $[0,\gamma[$, and h is increasing on $[\gamma,1]$.
As a consequence, for any $\alpha \in [0,\alpha_I^0]$ there exists $\beta_I(\alpha)$ such that

$$G\big(\alpha,\beta_I(\alpha)\big) - 2a\eta = 0 \tag{3.24}$$

and $G(\alpha,\beta) - 2a\eta > 0 \iff \beta < \beta_I(\alpha)$

c.2) Assume now that $h(\gamma) < 2a\eta < h\big(\beta^*(0)\big)$, which implies that $\alpha_e^0 = \alpha_e$. In this case, it follows that,

$$\text{if } \alpha < \alpha_e \,, \qquad G\big(\alpha,\beta^*(\alpha)\big) = h\big(\beta^*(\alpha)\big) > h(\beta_e) = 2a\eta \tag{3.25}$$

and thus $G(\alpha,\beta) > 2a\eta$ if $(\alpha,\beta) \in A$ and $\alpha < \alpha_e$.
Also, since $G(\alpha_e,\beta_e) = 2a\eta$ and G is strictly decreasing in β, it follows that $G(\alpha_e,\gamma) > 2a\eta$.
We thus infer from (3.23) that there exist $\alpha_I^0 \in \,]\alpha_e,\hat{\alpha}[$ such that:

$$G(\alpha,\gamma) = \begin{cases} > 2a\eta & \text{if } \alpha < \alpha_I^0 \\ < 2a\eta & \text{if } \alpha > \alpha_I^0 \\ 2a\eta & \text{if } \alpha = \alpha_I^0 \end{cases}$$

Now, in the same way that we obtained (3.25), we obtain that: If $\alpha > \alpha_e$ then $G\big(\alpha,\beta^*(\alpha)\big) < 2a\eta$.
This proves that for any $\alpha \in [\alpha_e,\alpha_I]$ there exists a unique $\beta_I(\alpha)$ with the requested properties.

Notice that in the cases (c.1)) (c.2)), $\beta_I(\alpha)$ satisfies the equality (3.24). From the implicit function theorem, since

$$\frac{\partial G}{\partial \beta}\big(\alpha,\beta_I(\alpha)\big) < 0 \qquad \text{and} \qquad \frac{\partial G}{\partial \alpha}\big(\alpha,\beta_I(\alpha)\big) < 0 \qquad \forall \alpha \in \,]\alpha_e^0,\alpha_I^0[$$

we arrive at the claimed result.

□

Proposition 3.12. *Assume that in the time interval $[t_1,t_2[$ with $0 \leq t_1 < t_2 \leq +\infty$, the solution $\big(\alpha(t),\beta(t)\big)$ of the system (2.42-2.44) is of class $\mathscr{C}^1$ and is on the curve Γ_4, i.e :*

$$\begin{cases} \alpha'(t) = \frac{S}{2}\dfrac{G\big(\alpha(t),\beta(t)\big) - 2a\eta}{h\big(\alpha(t)\big) - \eta} \\ G\big(\alpha(t),\beta(t)\big) = h\big(\beta(t)\big) \qquad \forall t \in [t_1,t_2] \\ \beta'(t) \leq z_{a,\lambda} \end{cases} \tag{3.26}$$

Then,

$$\begin{cases} \beta'(t) = \varphi\big(\beta(t)\big) \qquad \forall t \in [t_1,t_2] \\ \varphi\big(\beta(t)\big) \leq z_{\alpha,\lambda} \end{cases} \tag{3.27}$$

with

$$\begin{array}{rccl} \varphi: &]\beta^*(\widetilde{\alpha}),\beta^*(0)[& \to & \mathbb{R} \\ & \beta & \mapsto & \frac{S}{2}\,\frac{1}{h'(\beta)}\left(\dfrac{h(\beta)-h(\alpha)}{h^3(\alpha)\int_\alpha^\beta \frac{\mathrm{d}x}{h^3(x)}}\right)\left(\dfrac{h(\beta)-2a\eta}{h(\alpha)-\eta}\right) \end{array}$$

where $\alpha = \big(\beta^*\big)^{-1}(\beta)$.

Conversely, if $\beta \in \mathscr{C}^1\big([t_1,t_2]\big)$, $\beta\big([t_1,t_2]\big) \subset \,]\beta^*(\widetilde{\alpha}),\beta^*(0)[$ and satisfies (3.27), then $\big(\alpha(t),\beta(t)\big)$ satisfies (3.26) with $\alpha(t) = \big(\beta^*\big)^{-1}\big(\beta(t)\big)$.

Proof. Suppose that $\big(\alpha(t), \beta(t)\big)$ satisfy

$$G\big(\alpha(t), \beta(t)\big) = h\big(\beta(t)\big)$$

Differentiating in t we obtain

$$\frac{\partial G}{\partial \alpha}\big(\alpha(t), \beta(t)\big)\alpha'(t) + \left(\frac{\partial G}{\partial \beta}\big(\alpha(t), \beta(t)\big) - h'\big(\beta(t)\big)\right)\beta'(t) = 0$$

From (2.42) and (3.6), we have

$$\beta'(t) = \left(\frac{h(\alpha) - G(\alpha, \beta)}{h^3(\alpha)\int_\alpha^\beta \frac{\mathrm{d}x}{h^3(x)}}\right) \cdot \left(\frac{S}{2} \cdot \frac{G(\alpha, \beta) - 2a\eta}{h(\alpha) - \eta}\right) \cdot \left(\frac{1}{\frac{h(\beta) - G(\alpha, \beta)}{h^3(\beta)\int_\alpha^\beta \frac{\mathrm{d}x}{h^3(x)}} - h'(\beta)}\right)$$

Since $G(\alpha, \beta) = h(\beta)$ we obtain

$$\beta'(t) = \left(\frac{h(\alpha) - h(\beta)}{h^3(\alpha)\int_\alpha^\beta \frac{\mathrm{d}x}{h^3(x)}}\right) . \left(\frac{S}{2} . \frac{h(\beta) - 2a\eta}{h(\alpha) - \eta}\right) . \left(\frac{1}{-h'(\beta)}\right)$$

Therefore (3.27).

Conversely, assume that $\beta(t)$ is a solution of (3.27) and $\alpha(t) = \big(\beta^*\big)^{-1}\big(\beta(t)\big)$. We must prove that

$$\alpha'(t) = \frac{S}{2}\,\frac{G(\alpha, \beta) - 2a\eta}{h(\alpha) - \eta}$$

Differentiating $\alpha = \big(\beta^*\big)^{-1}(\beta)$ we have

$$\alpha'(t) = \frac{1}{\big(\beta^*\big)'\big(\alpha(t)\big)}\beta'(t)$$

Moreover when we differentiate the equality $G\big(\alpha, \beta^*(\alpha)\big) = h(\beta^*)$ with respect to α we have

$$\big(\beta^*\big)'(\alpha) = \frac{\frac{\partial G}{\partial \alpha}(\alpha, \beta^*)}{h'(\beta^*)} \qquad \left(\text{because } \frac{\partial G}{\partial \beta}(\alpha, \beta^*) = 0\right)$$

From (3.27) and (3.6) the result follows. □

We define the function corresponding to (2.42) and (2.44), which hold for $z_{a,\lambda} < \beta' \leq aS$,

$$\begin{array}{rccl} \phi: & \bar{A} & \to & \mathbb{R}^2 \\ & (\alpha, \beta) & \mapsto & \big(\phi_1(\alpha, \beta), \phi_2(\alpha, \beta)\big) = \left(\frac{S}{2}\frac{G(\alpha,\beta) - 2a\eta}{h(\alpha) - \eta}, \frac{S}{2(1-\lambda)}\big(\frac{G(\alpha,\beta)}{h(\beta)} - 2a\lambda\big)\right) \end{array}$$

where $\bar{A}$ is the closure of A, with A defined in (3.22). For all $(\alpha, \beta) \in \partial A$, we define $\nu(\alpha, \beta)$ as the external normal vector to ∂A. Since $G(\alpha, \beta) > h(\beta)$ on $\mathring{A}$ (the interior of A) and $\lambda \leq \frac{1}{2a}$, we have $\phi_2(\alpha, \beta) > 0 \quad \forall(\alpha, \beta) \in \mathring{A}$. This gives

$$\phi \cdot \nu < 0 \qquad \text{on } \Gamma_2 \tag{3.28}$$

Moreover, from Lemma 3.11 we infer that $\phi_1(\alpha, \beta) < 0$ if (α, β) is sufficiently close to Γ_3, implying that

$$\phi \cdot \nu < 0 \qquad \text{if } (\alpha, \beta) \text{ is close to } \Gamma_3 \tag{3.29}$$

In what regards the sign of $\phi \cdot \nu$ when $(\alpha, \beta) \in \Gamma_4$, we have $\nu = \frac{1}{\sqrt{1 + \big((\beta^*)'\big)^2}}\begin{pmatrix} -(\beta^*)' \\ 1 \end{pmatrix}$ and $G(\alpha, \beta^*) = h(\beta^*)$.

We immediately obtain that

$$\phi \cdot \nu = \frac{1}{\sqrt{1 + \big((\beta^*)'\big)^2}}\left[-\frac{S}{2}\frac{h(\beta) - 2a\eta}{h(\alpha) - \eta}\big(\beta^*\big)' + z_{a,\lambda}\right] \qquad \text{if } (\alpha, \beta) \in \Gamma_4 \text{ and } a > \frac{1}{2} \tag{3.30}$$

Notice that if $\lambda = \frac{1}{2a}$ we have $z_{a,\lambda} = 0$.

Since $(\beta^*)' < 0$ and $h(\alpha) - \eta > 0$, we get the following proposition :

Proposition 3.13. *Assume $\lambda = \frac{1}{2a}$. Then we have :*

a) if $2a\eta \geq h\big(\beta^*(0)\big)$, then $\phi \cdot \nu < 0$ on Γ_4

b) if $2a\eta \leq h\big(\gamma\big)$, then $\phi \cdot \nu > 0$ on Γ_4

c) if $h(\gamma) < 2a\eta < h\big(\beta^*(0)\big)$, then a stationary solution exist (α_e, β_e). Further, $\phi \cdot \nu < 0$ if $\beta < \beta_e$ and $\phi \cdot \nu > 0$ if $\beta > \beta_e$.

3.3.2. *Results for the global existence of solution.*
In this part we carry out the analysis with a specific choice of λ, namely

$$\lambda = \frac{1}{2a} \tag{3.31}$$

which implies $z_{a,\lambda} = 0$. We will prove that the solution $\big(\alpha(t), \beta(t)\big)$ can be :

- either global in time and, in this case, it converges to the equilibrium state,
- or the solution exists only in a finite time interval $[0, T_1]$, and in this case, we have $\alpha(T_1) = 0$ (corresponding to the reformation boundary leaving the calculation domain) or $\big(\alpha(T_1), \beta(T_1)\big) = (\gamma, \gamma)$ (corresponding to the case in which the pressurized region gradually reduces to a single point and then disappears, rendering the model invalid after that time).

This result is structured into our main theorem, which reads as follows:

Theorem 3.14. *Assume hypotheses (H1), (H2), (3.15) and (3.18), together with the choice (3.31) and let $(\alpha_0, \beta_0) \in \mathring{A}$. Then,*

***Case** 1. if $2ah(0) > 2a\eta \geq h\big(\beta^*(0)\big)$ there exists $T_1 > 0$ such that the solution of (2.42-2.44) exists in A and is unique for $t \in [0, T_1]$ and $\alpha(T_1) = 0$*

***Case** 2. if $h(\gamma) \leq 2a\eta < h\big(\beta^*(0)\big)$ the solution of (2.42-2.44) exists exists in A and is unique for $t \in [0, +\infty[$, and $\big(\alpha(t), \beta(t)\big) \xrightarrow[t \to +\infty]{} (\alpha_e, \beta_e)$*

***Case** 3. if $h(\gamma) > 2a\eta > 0$ there exists $T_2 > 0$ such that the solution of (2.42-2.44) exists exists in A and is unique for $t \in [0, T_2]$ and $\big(\alpha(T_2), \beta(T_2)\big) = (\gamma, \gamma)$.*

Proof. It is obvious that there exist a time $T_0 > 0$ such that for all $t \in [0, T_0[$, the solution $\big(\alpha(t), \beta(t)\big)$ of the problem (2.42-2.44) is a solution of the following system:

$$\begin{cases} \alpha' = \phi_1(\alpha, \beta) \\ \beta' = \phi_2(\alpha, \beta) \\ \alpha(0) = \alpha_0 \\ \beta(0) = \beta_0 \end{cases} \tag{3.32}$$

We note T_0 the largest time for which the solution of (3.32) exists and is in $\mathring{A}$.

From (3.13) and from the description of (3.14) and of (3.22) in A, we have $\beta'(t) > 0$.

Case 1. a) If $2ah(0) > 2a\eta \geq G(0, \gamma)$ then from Lemma 3.11.a) we have $\alpha'(t) < 0$. Thus, $\big(\alpha(t), \beta(t)\big)$ does not intersect Γ_4, since $\phi \cdot \nu < 0$ there (from Proposition 3.13). Then the model never switches to equation 2.43 and the proof is claimed with $T_1 = T_0$.

b) If $h\big(\beta^*(0)\big) \leq 2a\eta < G(0, \gamma)$ then there exist an isoline Z, defined in Lemma 3.11, and we have

$$\begin{cases} \alpha' > 0 & \text{if } (\alpha, \beta) \in A_\ell \\ \alpha' < 0 & \text{if } (\alpha, \beta) \in A_r \end{cases}$$

We infer from this that $T_1 = T_0$, because $\big(\alpha(t), \beta(t)\big)$ can intersect Z coming from A_ℓ to A_r but not the other way around, since in Z the tangent to the solution curve is vertical and oriented upwards. Moreover, $\big(\alpha(t), \beta(t)\big)$ does not intersect Γ_4 because $\phi \cdot \nu < 0$ on Γ_4.

Case 2. If $h\big(\beta^*(0)\big) > 2a\eta \geq h(\gamma)$, then we have a stationary solution (α_e, β_e) and an isocline Z as defined in Lemma 3.11. Two cases must be considered:

a) If $(\alpha_0, \beta_0) \in A_r$, the graph of $\big(\alpha(t), \beta(t)\big)$ does not intersect Z (for the same reason as in case 1b)), nor the trajectory of $\{\ (\alpha, \beta) \in \Gamma_4 \mid \alpha > \alpha_e\ \}$, nor the line $\{\ \beta = \gamma\ \}$. This implies that $\big(\alpha(t), \beta(t)\big)$ exists for all $t \in [0, +\infty[$ (thus, $T_0 = +\infty$), that $\big(\alpha(t), \beta(t)\big) \in \mathring{A}_r$, and that $\big(\alpha(t), \beta(t)\big) \xrightarrow[t \to +\infty]{} (\alpha_e, \beta_e)$.

b) If $(\alpha_0, \beta_0) \in A_\ell$, three cases must be considered:
   i) $\big(\alpha(t), \beta(t)\big) \in \mathring{A}_\ell$, $\forall t \geq 0$ and $\big(\alpha(t), \beta(t)\big) \xrightarrow[t\to+\infty]{} (\alpha_e, \beta_e)$
   ii) There exist $t_1 > 0$ such as $\big(\alpha(t_1), \beta(t_1)\big) \in Z$. Then the trajectory after this time t_1 is like in the case a)).
   iii) $T_0 < +\infty$, $\big(\alpha(T_0), \beta(T_0)\big) \in \Gamma_4$ and $\alpha(T_0) < \alpha_e$ (this happens if $\beta(t) > \beta_e$ for some t). In this case, $\beta(t)$ satisfies the differential equation 3.27 at least in the interval $]T_0, T_1[$ (so $\big(\alpha(t), \beta(t)\big)$ evolves on the part $\{\ \beta \geq \beta_e\ \}$ of the boundary Γ_4). Notice that $h(\beta_e) = 2a\eta$ and that if $\beta > \beta_e$ then $\varphi(\beta) < 0$ and $\varphi(\beta_e) = 0$, with α as defined in Propostion (3.12). This proves that $T_1 = +\infty$, that $\beta'(t) < 0$ for $t > T_0$, and that $\beta(t) \xrightarrow[t\to+\infty]{} \beta_e$. As a consequence, $\alpha(t) \xrightarrow[t\to+\infty]{} \alpha_e$, implying that $\big(\alpha(t), \beta(t)\big)$ will remain on the part $\{\ \beta \geq \beta_e\ \}$ of the boundary Γ_4 and that $\big(\alpha(t), \beta(t)\big) \xrightarrow[t\to+\infty]{} (\alpha_e, \beta_e)$.

Case 3. If $h(\gamma) > 2a\eta > 0$ we have that $\alpha'(t) > 0$ and that there is no isoline Z. In this case, $\big(\alpha(T_0), \beta(T_0)\big) \in \Gamma_4$ and thereafter the trajectory of $\big(\alpha(t), \beta(t)\big)$ is as in case iii) above, except that $\beta'(t) < 0$ for $t > T_0$. Moreover, there is no steady state, the trajectory of $\big(\alpha(t), \beta(t)\big)$ goes to (γ, γ) in a **finite** time.

□

Remark 3.15. *The choice $\lambda = \frac{1}{2a}$ may not be necessary for the result to hold. It was adopted to avoid the cumbersome calculations that result from carrying the model in its full generality. The main interest is in the case $a = 1$, for which computations with a CFD code* [2, 4] *suggest that in the steady state ($\beta' = 0$) the rupture thickness satisfies $\theta(\beta^+) = \frac{1}{2}$. This is reproduced by the choice $\lambda = \frac{1}{2a}$.*

4. Numerical results

It is interesting to illustrate the behavior of the proposed model with a few numerical results. A more exhaustive numerical assessment together with details of approximation by finite volumes can be found in a companion article [4].

We consider here the problem (2.38)-(2.41), with $h(x) = 1 + (2x - 1)^2$ and $S = 1$. The initial conditions are such that $\Omega^+ =]\alpha(t), \beta(t)[$ and we denote $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$. Further, we impose the conditions $\theta_0(x)h(x) = \eta = \theta_{in}(t)h(0)$ so that the model is equivalent to the algebraic-differential system (2.42)-(2.45).

We take $a\,\eta = 0.52$ so that a stationary solution exists, and we solve the system in time using an explicit scheme. Three sets of parameters are adopted: (a) $a = \frac{1}{2}$, $\lambda = 1$; (b) $a = 1$, $\lambda = \frac{1}{2}$; and (c) $a = 1$, $\lambda = 0$. The first case (a) corresponds to the Elrod-Adams model and is included for comparison.

The evolution of α and β as functions of time is shown in Fig. 6. The differences between the new model and the Elrod-Adams model are quite significant. For the new model, the effect of the choice of λ is best observed in the evolution of the rupture boundary, $\beta(t)$. The trajectory of the system in the $\alpha - \beta$ plane is shown in Fig. 7. It is exactly as is predicted in the theory, approaching Γ_4 and tending to (α_e, β_e). Also shown there is the field θ at $t = 4.8$, which for the new model is discontinuous at both α and β, while the Elrod-Adams model only yields discontinuity at α. The graph of θ is best understood in terms of the oil-film thickness, which is depicted in Fig. 8 Finally, the saturation variable θ and the oil-film thickness at the stationary state are shown in Fig. 9. The location of the reformation and rupture boundaries, and thus the pressure field, is the same for the cases (a), (b) and (c). In fact, the steady state produced by the new model coincides with that of the Elrod-Adams model whenever $\lambda \leq \frac{1}{2a}$, provided that the same inflow flux ($J = a\,S\,\eta$) is imposed to the two models. Of course, in the cavitated regions it is not the values of θ that will coincide, but the values of $a\,\theta$ instead. The approach to the steady state, on the other hand, is quite different, as are different the predictions in all transient situations in general.

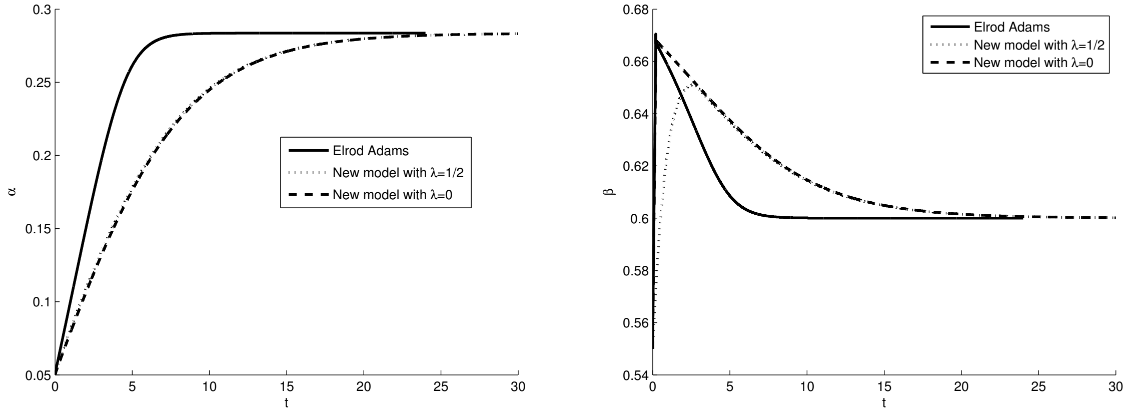

FIGURE 6. Graphs of $\alpha(t)$ (left) and $\beta(t)$ (right), for the three models considered.

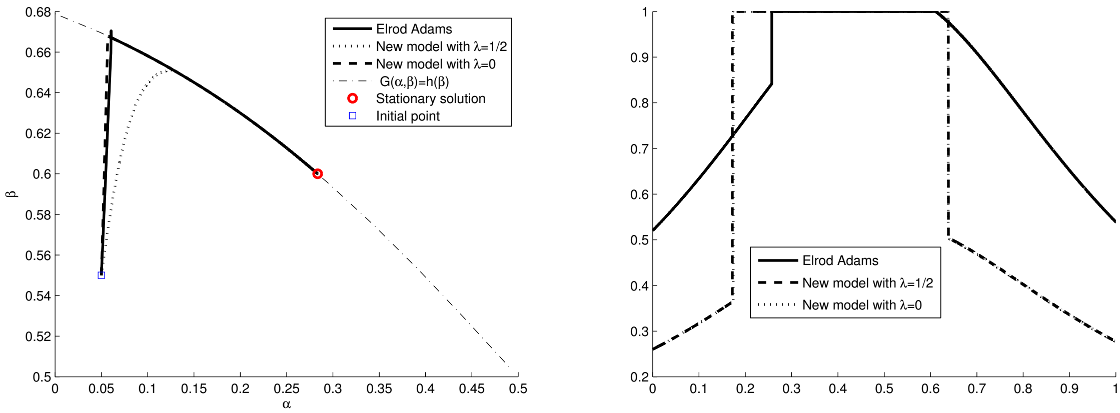

FIGURE 7. Evolution of the system in the phase plane $\alpha - \beta$ (left). Profiles of θ vs. x at time $t = 4.8$ (right).

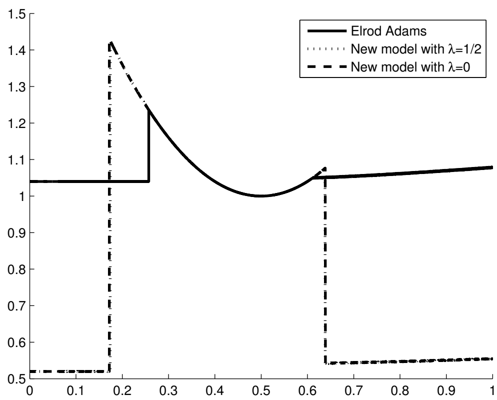

FIGURE 8. Film thickness at $t = 4.8$.

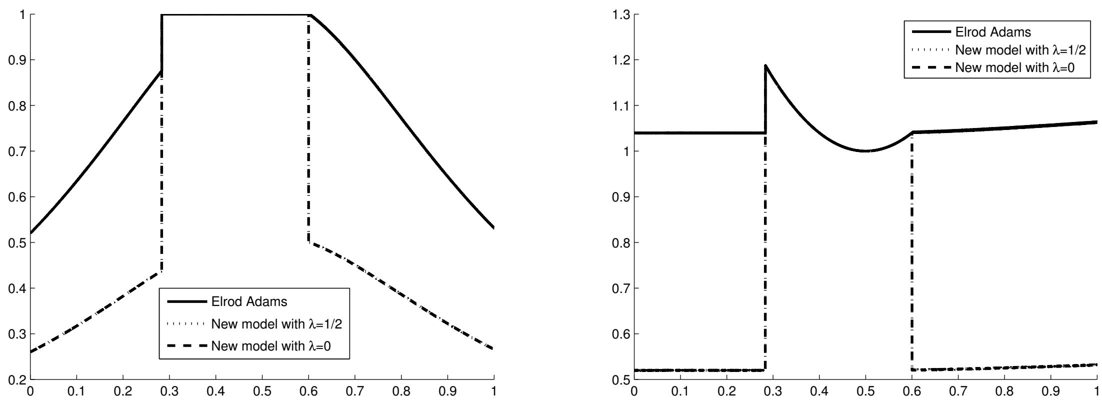

FIGURE 9. Profiles of θ (left) and of the film thickness (right) at the stationary solution.

5. CONCLUSIONS

A generalization of the Elrod-Adams $p - \theta$ model has been proposed that allows for more flexibility in the translation velocity of the lubricant in the cavitated region. To achieve this, a naive generalization was first analyzed and its loss of uniqueness explained. An additional condition was then proposed at the rupture boundary that allows to recover uniqueness of solution. The resulting model was mathematically analyzed in a simple one-dimensional setting with just one connected pressurized region, and the existence of a unique solution globally in time was proved.

Work is under progress to assess this model numerically for the piston-ring/liner contact problem in the textured case, with encouraging results (the first non-trivial tests can be found in [4]). The challenge that lies ahead is the reformulation of the proposed model as a global differential problem qualitatively similar to the $p-\theta$ model. This will allow to avoid the tracking of the interface and to tackle realistic situations with multiple, time–evolving pressurized regions in two dimensions.

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