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Interpolation estimate for a finite-element space with embedded discontinuities

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We consider a recently proposed finite-element space that consists of piecewise affine functions with discontinuities across a smooth given interface Γ (a curve in two dimensions, a surface in three dimensions). Contrary to existing extended finite element methodologies, the space is a variant of the standard conforming $\mathbb{P}_1$ space that can be implemented element by element. Further, it neither introduces new unknowns nor deteriorates the sparsity structure. It is proved that, for u arbitrary in $W^{1,p}(\Omega \setminus \Gamma) \cap W^{2,s}(\Omega \setminus \Gamma)$, the interpolant $\mathcal{I}_h u$ defined by this new space satisfies

$$\|u - \mathcal{I}_h u\|_{L^q(\Omega)} \leqslant C \left[h^{1+\frac{1}{q}-\frac{1}{p}} |u|_{W^{1,p}(\Omega\setminus\Gamma)} + h^2 |u|_{W^{2,s}(\Omega\setminus\Gamma)} \right],$$

where h is the mesh size, $\Omega \subset \mathbb{R}^d$ is the domain, $p > d$, $p \geqslant q$, $s \geqslant q$ and standard notation has been adopted for the function spaces. This result proves the good approximation properties of the finite-element space as compared to any space consisting of functions that are continuous across Γ, which would yield an error in the $L^q(\Omega)$-norm of order $h^{\frac{1}{q}-\frac{1}{p}}$. These properties make this space especially attractive for approximating the pressure in problems with surface tension or other immersed interfaces that lead to discontinuities in the pressure field. Furthermore, the result still holds for interfaces that end within the domain, as happens for example in cracked domains.

Keywords: finite element; interpolation; interface; discontinuous pressure; cracked domain; surface tension.

1. Introduction

In Eulerian, fixed grid methods for fluid mechanics, interfaces that exist within the domain are not followed by the mesh. This creates difficulties in the approximation of those variables that are discontinuous at the interface, a typical example being the pressure in multiphase flow with different viscosities for the phases and/or surface tension effects, or when the interface between the phases behaves as an elastic membrane.

Let Γ be an interface (embedded in some domain Ω) at which some function $u \in L^2(\Omega)$ is discontinuous, and let u_h be its $L^2(\Omega)$-projection onto some function space V_h that consists of functions that are *continuous* at Γ. Then, no matter how smooth u is outside Γ, the approximation order is (Ganesan *et al.*, 2007; Gross & Reusken, 2007)

$$\|u - u_h\|_{L^2(\Omega)} \simeq C h^{\frac{1}{2}}, \tag{1.1}$$

where h is the mesh size and C, here and later, denotes a generic constant. If Γ is not aligned with the mesh, then all standard finite-element spaces, either continuous or discontinuous across interelement boundaries, suffer from this poor approximation order. The reason for this is that, in this situation, the discontinuity will pass through the element interiors at which standard finite-element interpolants are continuous (typically polynomial).

Some attempts have been made in recent years to devise spaces with improved approximation properties. Gross & Reusken (2007) recently proposed the adoption of an extended finite element (XFEM) (Belytschko *et al.*, 2001) enrichment of the pressure space, incorporating functions that are discontinuous at Γ, as had also been proposed by Minev *et al.* (2003). They obtained improved ($O(h^2)$) convergence order at the expense of the well-known pitfalls of the XFEM methodology, namely, the ill-conditioning of the system matrix due to approximate linear dependence of the basis and the introduction of new unknowns that depend on the location of the interface, thus requiring the code to completely rebuild the linear system structure for each interface location.

In this article we analyse an alternative approach due to Ausas *et al.* (2010), that has the following interesting properties: (a) it is a modification of the standard continuous $\mathbb{P}_1$ finite-element space, with exactly the same unknowns and connectivity; (b) the modified basis functions can be computed at the element level, making its implementation straightforward in existing codes; (c) its approximation order (in the $L^2(\Omega)$-norm) is $h^{\frac{3}{2}}$, which is lower than that of the XFEM space but is one order higher than that of any standard space; and (d) its approximation order is already higher than the velocity approximation order of the minielement (Arnold *et al.*, 1984) or of stabilized $\mathbb{P}_1/\mathbb{P}_1$ formulations (Hughes *et al.*, 1986; Franca & Hughes, 1988), both of order at most h, so that using this space for the pressure has no adverse effect on the overall velocity–pressure approximation. In this last regard we remind the reader that the natural error measure for Stokes or Navier–Stokes problems is the sum of the $H^1(\Omega)$-error for velocity and the $L^2(\Omega)$-error for pressure. The result is established here in the more general $L^q(\Omega)$-setting, where $q \geqslant 1$, because this entails no additional difficulties.

2. Description of the finite-element space

We first analyse the case in which Γ does not end within the domain, and later discuss the modification required in the case in which Γ is crack-like.

Let V_h be the standard conforming $\mathbb{P}_1$ finite-element space defined on a triangulation $\mathcal{T}_h$ of Ω. The idea behind the space proposed by Ausas *et al.* (2010), denoted by W_h from now on, is quite simple. To begin with, in those elements that are not cut by Γ the space W_h coincides with V_h and the basis functions are simply those of the conforming $\mathbb{P}_1$ finite-element space. In the elements that intersect Γ, each edge E (in both two dimensions and three dimensions the segments joining nodes of $\mathcal{T}_h$ are referred to as *edges*) cut by Γ is divided into two parts, E^+ and E^-, according to the side of Γ on which the part lies. The functions in W_h are then defined by their values at the original nodes of the mesh plus the new nodes created at the intersections of the mesh edges with Γ. The functions in W_h are bi-valued at these new nodes, but their values there are not degrees of freedom that are available for interpolation. Instead, the value on each side is defined as being the same as that of the (unique) node of the original triangulation that lies on the same edge and on the same side. In the rest of this section we

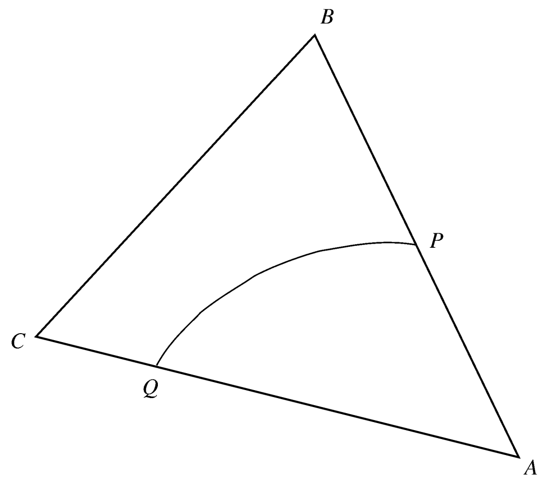

FIG. 1. Partition of a single element into subparts following the interface *PQ*.

recall the corresponding basis functions in two dimensions, stressing some properties that are essential for our proof of the interpolation estimate. More details, together with the three-dimensional version of the basis functions, can be found in Ausas *et al.* (2010).

We assume that no edge of the triangulation is cut twice by Γ. Consider the element K to be the triangle ABC, which is cut by Γ at the points P and Q. The (possibly curved) segment PQ divides K into two *subparts*, namely, the curved triangle APQ and the curved quadrilateral $BCQP$ (see Fig. 1).

REMARK 2.1 Though in most of the figures hereafter the piece of interface $\Gamma \cap K$ is shown as a straight segment for ease of interpretation, the reader should keep in mind that Γ has neither been assumed to be a polygon/polyhedron nor approximated by one. The usual mapping techniques to define $\mathbb{P}_k$ or $\mathbb{Q}_k$ interpolants on curved elements are adopted, and it is tacitly assumed that $\Gamma \cap K$ is sufficiently well behaved for the mappings to be well defined. This will always be the case if the mesh size is much smaller than the local radius of curvature of $\Gamma \cap K$. Hereafter, we also omit the word 'curved' when referring to a piece of Γ, calling 'curved triangles' simply 'triangles', etc.

Let us arbitrarily denote the triangle APQ as the 'plus' side of Γ and the quadrilateral $BCQP$ as the 'minus' side. One can choose here to either adopt a $\mathbb{Q}_1$ interpolation in $BCQP$ or subdivide the quadrilateral into two triangles, BCP and CQP. In what follows we adopt the latter option.

Three nodal basis functions, φ_A, φ_B and φ_C, can then be defined satisfying the following properties:

- P1 $\varphi_\alpha(\beta) = 1$ if $\alpha = \beta$ and $\varphi_\alpha(\beta) = 0$ otherwise, where α and β can take the values A, B or C;
- P2 $\varphi_A + \varphi_B + \varphi_C = 1$;
- P3 at any point of K, the values of φ_A, φ_B and φ_C lie between 0 and 1;
- P4 though not necessary for our interpolation result, it is also true that, when combined with the corresponding basis functions of the neighbour elements, the resulting functions are continuous everywhere outside Γ.

Analogous properties are satisfied by the four basis functions proposed by Ausas *et al.* (2010) for three-dimensional interpolation.

The basis functions are defined to be piecewise $\mathbb{P}_1$ inside each of the subtriangles APQ, BCP and CQP. It only remains to define their values at the vertices of the subtriangles, that is at the points A, B, C, P and Q, but, since they are discontinuous at Γ, two values (plus and minus) are given at the points P and Q. These values are

$$\varphi_A(A) = 1, \quad \varphi_B(A) = 0, \quad \varphi_C(A) = 0, \tag{2.1}$$

$$\varphi_A(B) = 0, \quad \varphi_B(B) = 1, \quad \varphi_C(B) = 0, \tag{2.2}$$

$$\varphi_A(C) = 0, \quad \varphi_B(C) = 0, \quad \varphi_C(C) = 1, \tag{2.3}$$

$$\varphi_A(P^+) = 1, \quad \varphi_B(P^+) = 0, \quad \varphi_C(P^+) = 0, \tag{2.4}$$

$$\varphi_A(P^-) = 0, \quad \varphi_B(P^-) = 1, \quad \varphi_C(P^-) = 0, \tag{2.5}$$

$$\varphi_A(Q^+) = 1, \quad \varphi_B(Q^+) = 0, \quad \varphi_C(Q^+) = 0, \tag{2.6}$$

$$\varphi_A(Q^-) = 0, \quad \varphi_B(Q^-) = 0, \quad \varphi_C(Q^-) = 1. \tag{2.7}$$

From the first three lines above we verify that property P1 is satisfied. Since the sum of the three functions is 1 at all vertices, their sum is the constant function equal to 1 and property P2 is satisfied. The interpolation being piecewise $\mathbb{P}_1$, the extrema take place at the vertices and property P3 follows from direct inspection. These basis functions are depicted in Figure 2.

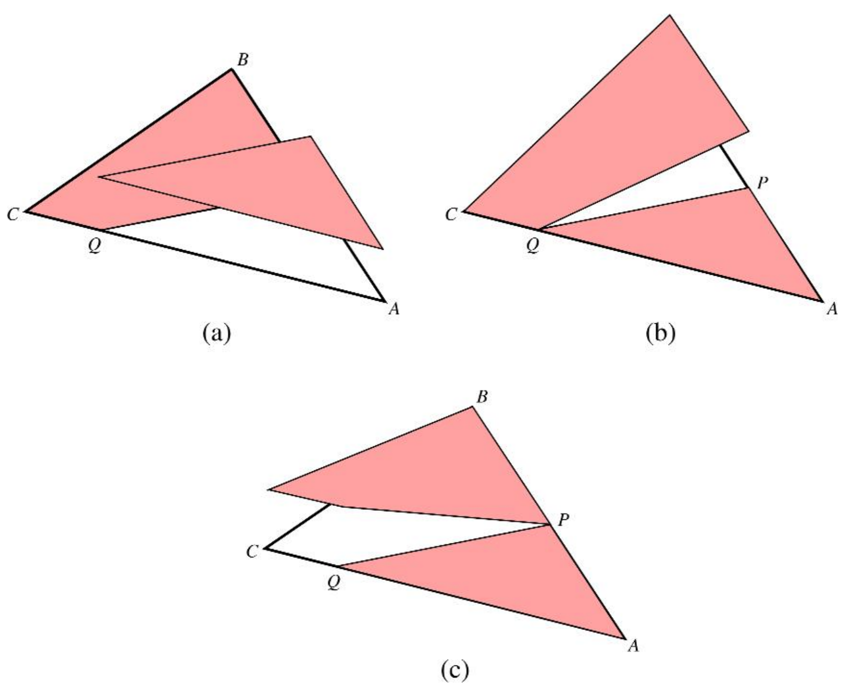

FIG. 2. Basis functions for the new finite-element space: (a) φ_A, (b) φ_B and (c) φ_C.

3. Interpolation estimate

Let $\Omega \subset \mathbb{R}^d$ be an open bounded polygonal domain, and let Γ be a smooth curve in Ω (a smooth surface if $d = 3$) that does not end within the domain. Let us define a 'regular' triangulation $\mathcal{T}_h$ of Ω to be a partition into simplices that has no hanging nodes (conforming triangulation) and that, in addition, satisfies the following hypotheses.

- H1 Each edge (segment that adjoins two vertices) of each simplex is cut at most once by Γ.
- H2 The interface Γ does not pass by any vertex. This is a technical assumption that is needed for the interpolant to be defined. The interpolation estimate does not deteriorate when the distance from Γ to one of the vertices tends to 0.
- H3 For all $K \in \mathcal{T}_h$ the mesh size h satisfies $ch \leqslant \rho(K) \leqslant \mathrm{diam}(K) \leqslant h$, where ρ is the radius of the inscribed circle/sphere, and thus $c^d h^d \leqslant \mathrm{meas}(K) \leqslant h^d$, with $0 < c < 1$ being independent of h. Note that, in particular, the triangulation needs to be quasiuniform.
- H4 The triangulation is such that, for each K and for each node I of K, the subpart that contains I (as defined in Fig. 1) is star-shaped with respect to I. This is equivalent to requiring 'visibility' of $\Gamma \cap K$ from I, that is, for each $x \in \Gamma \cap K$ the straight segment $\overline{xI}$ only intersects Γ at x.

NOTATION 3.1 From now on, for any measurable set ω we will define

$$|\omega| = \mathrm{meas}(\omega).$$

For $u \in W^{1,p}(\Omega \setminus \Gamma)$ (with $p > d$, so that functions are continuous in $\Omega \setminus \Gamma$) let $\mathcal{I}_h u$ be the interpolant of u in W_h, defined by

$$\mathcal{I}_h u(x) = \sum_{i=1}^{N_V} u(X_i)\varphi_i(x), \tag{3.1}$$

where $x \in \Omega$, N_V is the number of vertices and φ_i is the basis function associated to vertex (node) number i, which is located at $X_i \in \mathbb{R}^d$.

This section is devoted to the proof of the following estimate.

THEOREM 3.2 Let p, q and s be three given real numbers satisfying $p > d$, $1 \leqslant q \leqslant p$ and $s \geqslant q$. Then there exists a constant C such that, for all regular triangulations $\mathcal{T}_h$ of Ω, and for all $u \in W^{1,p}(\Omega \setminus \Gamma) \cap W^{2,s}(\Omega \setminus \Gamma)$, we have

$$\|u - \mathcal{I}_h u\|_{L^q(\Omega)} \leqslant C\left[h^{1+\frac{1}{q}-\frac{1}{p}}|u|_{W^{1,p}(\Omega\setminus\Gamma)} + h^2|u|_{W^{2,s}(\Omega\setminus\Gamma)}\right]. \tag{3.2}$$

For the proof we first establish some lemmata.

LEMMA 3.3 Let ω be a bounded open set in $\mathbb{R}^d$. Let $p > d$ and let $y \in \overline{\omega}$ be a point such that ω is star-shaped with respect to y. Then, for all $w \in L^p(\omega)$, we have

$$\int_\omega \left[\int_0^1 |w(y + t(x-y))|\,\mathrm{d}t\right]^p \mathrm{d}x \leqslant \left(\frac{p}{p-d}\right)^p \|w\|^p_{L^p(\omega)}. \tag{3.3}$$

Proof. Note first that the integral on the left-hand side makes sense since $y + t(x - y) \in \omega$ for all $t \in (0, 1)$ because ω is star-shaped with respect to y. Without loss of generality, we take $y = 0$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. We multiply and divide by t^α, where $\alpha = \frac{(p-1)d}{p^2}$, to obtain

$$\begin{aligned}\int_\omega \left[\int_0^1 |w(tx)|\mathrm{d}t\right]^p \mathrm{d}x &= \int_\omega \left[\int_0^1 |w(tx)| t^\alpha t^{-\alpha}\,\mathrm{d}t\right]^p \mathrm{d}x \\ &\leqslant \int_\omega \left(\int_0^1 |w(tx)|^p t^{\alpha p}\,\mathrm{d}t\right)\left(\int_0^1 t^{-\alpha q}\,\mathrm{d}t\right)^{\frac{p}{q}} \mathrm{d}x \\ &= \left(\frac{1}{1-\alpha q}\right)^{\frac{p}{q}} \int_\omega \int_0^1 |w(tx)|^p \; t^{\alpha p}\,\mathrm{d}t\,\mathrm{d}x \\ &= \left(\frac{p}{p-d}\right)^{p-1} \int_\omega \int_0^1 |w(z)|^p t^{\alpha p - d}\mathrm{d}t\,\mathrm{d}z \\ &= \left(\frac{p}{p-d}\right)^p \|w\|^p_{L^p(\omega)},\end{aligned}$$

where we have used Hölder's inequality in one dimension and the change of variables $z = tx$, implying that $\mathrm{d}t\,\mathrm{d}z = t^d\,\mathrm{d}t\,\mathrm{d}x$. □

LEMMA 3.4 Under the same hypotheses as Lemma 3.3 we have for all $1 \leqslant q \leqslant p$ that

$$\|w - w(y)\|_{L^q(\omega)} \leqslant \left(\frac{p}{p-d}\right) \mathrm{diam}(\omega)|\omega|^{\frac{1}{q}-\frac{1}{p}}|w|_{W^{1,p}(\omega)} \tag{3.4}$$

whenever $w \in W^{1,p}(\omega)$.

Proof. The proof follows a density argument. For $w \in C^1(\omega) \cap W^{1,p}(\omega)$ and for all $x \in \omega$ we have

$$w(x) - w(y) = \int_0^1 \nabla w(y + t(x-y)) \cdot (x-y)\mathrm{d}t,$$

implying that

$$\|w - w(y)\|^q_{L^q(\omega)} \leqslant \int_\omega \left|\int_0^1 \nabla w(y + t(x-y)) \cdot (x-y)\mathrm{d}t\right|^q \mathrm{d}x.$$

From Lemma 3.3 and Hölder's inequality with $r = \frac{p}{q}$ and $s = \frac{p}{p-q}$ (if $p = q$ then this is, of course, unnecessary) we then get

$$\int_\omega \left|\int_0^1 \nabla w(y + t(x-y)) \cdot (x-y)\mathrm{d}t\right|^q \mathrm{d}x \leqslant \left(\frac{p}{p-d}\right)^q \mathrm{diam}(\omega)^q |\omega|^{1-\frac{q}{p}} \|\nabla w\|^q_{L^p(\omega)}. \qquad \square$$

LEMMA 3.5 Let K be a simplex of $\mathcal{T}_h$ of vertices $\{X_i\}_{i=1,\dots,d+1}$ that is crossed by the interface Γ. Let $w \in W^{1,p}(K \setminus \Gamma)$ (with $p > d$), and let $\mathcal{I}_K w(x) = \sum_{i=1}^{d+1} w(X_i)\varphi_i(x)$ be its local interpolant. Then, for all $1 \leqslant q \leqslant p$, we have

$$\|w - \mathcal{I}_K w\|_{L^q(K)} \leqslant \frac{p(d+1)}{p-d} h^{1+\frac{d}{q}-\frac{d}{p}} |w|_{W^{1,p}(K\setminus\Gamma)}. \tag{3.5}$$

Proof. We start from the straightforward estimate

$$\|w - \mathcal{I}_K w\|_{L^q(K)} = \left\| \sum_{i=1}^{d+1} [w - w(X_i)]\varphi_i \right\|_{L^q(K)} \leqslant \sum_{i=1}^{d+1} \|w - w(X_i)\|_{L^q(K_i)},$$

where K_i denotes the support of φ_i. Now note that, by the construction of the basis functions, K_i is the connected component of $\overline{K}$ (connected in the sense that it lies on one side of Γ) that contains the vertex X_i, and K_i is star-shaped with respect to X_i by hypothesis H4. We thus apply Lemma 3.4 and hypothesis H3 to obtain

$$\|w - \mathcal{I}_K w\|_{L^q(K)} \leqslant \left(\frac{p}{p-d}\right) h^{1+\frac{d}{q}-\frac{d}{p}} \sum_{i=1}^{d+1} |w|_{W^{1,p}(K_i)},$$

from which the result follows. □

We also recall the following classical estimate for $\mathbb{P}_1$ elements that holds for all elements that are not intersected by Γ.

LEMMA 3.6 There exists $c_s > 0$ such that, for all $K \in \mathcal{T}_h$ and all $w \in W^{2,s}(K)$, we have

$$\|w - \mathcal{I}_K w\|_{L^q(K)} \leqslant c_s h^{2+\frac{d}{q}-\frac{d}{s}} |w|_{W^{2,s}(K)}. \tag{3.6}$$

We now proceed to the proof of Theorem 3.2.

Proof of Theorem 3.2. We begin by decomposing the mesh $\mathcal{T}_h$ into the subset $\mathcal{R}_h$ of elements *not intersected* by Γ, and the subset $\mathcal{S}_h$ of elements *intersected* by Γ. Due to the quasiuniformity of the mesh we know that the number of elements in $\mathcal{R}_h$ is $N_{\mathcal{R}_h} \leqslant ch^{-d}$, while the regularity of Γ ensures that $N_{\mathcal{S}_h} \leqslant ch^{-d+1}$, where c does not depend on h. We will estimate the two terms in

$$\begin{aligned}\|u - \mathcal{I}_h u\|_{L^q(\Omega)} &= \left(\sum_{K\in\mathcal{T}_h} \|u - \mathcal{I}_K u\|^q_{L^q(K)} \right)^{\frac{1}{q}} \\ &\leqslant \left(\sum_{K\in\mathcal{R}_h} \|u - \mathcal{I}_K u\|^q_{L^q(K)} \right)^{\frac{1}{q}} + \left(\sum_{K\in\mathcal{S}_h} \|u - \mathcal{I}_K u\|^q_{L^q(K)} \right)^{\frac{1}{q}}.\end{aligned} \tag{3.7}$$

The first term consists of classical $\mathbb{P}_1$ elements, so that from Lemma 3.6 we have

$$\begin{aligned}\sum_{K\in\mathcal{R}_h}\|u-\mathcal{I}_K u\|^q_{L^q(K)} &\leqslant c_s^q h^{2q+d-\frac{dq}{s}}\sum_{K\in\mathcal{R}_h}|u|^q_{W^{2,s}(K)}\\ &\leqslant c_s^q h^{2q+d-\frac{dq}{s}} N_{\mathcal{R}_h}^{1-\frac{q}{s}}\left(\sum_{K\in\mathcal{R}_h}|u|^s_{W^{2,s}(K)}\right)^{\frac{q}{s}}\\ &\leqslant c^{1-\frac{q}{s}}c_s^q h^{2q}|u|^q_{W^{2,s}(\Omega\setminus\Gamma)}.\end{aligned} \tag{3.8}$$

Turning now to the second term in (3.7), we have from Lemma 3.5 that

$$\begin{aligned}\sum_{K\in\mathcal{S}_h}\|u-\mathcal{I}_K u\|^q_{L^q(K)} &\leqslant \left(\frac{p(d+1)}{p-d}\right)^q h^{q+d-\frac{dq}{p}}\sum_{K\in\mathcal{S}_h}|u|^q_{W^{1,p}(K\setminus\Gamma)}\\ &\leqslant \left(\frac{p(d+1)}{p-d}\right)^q h^{q+d-\frac{dq}{p}} N_{\mathcal{S}_h}^{1-\frac{q}{p}}\left(\sum_{K\in\mathcal{S}_h}|u|^p_{W^{1,p}(K\setminus\Gamma)}\right)^{\frac{q}{p}}\\ &\leqslant c^{1-\frac{q}{p}}\left(\frac{p(d+1)}{p-d}\right)^q h^{q+1-\frac{q}{p}}|u|^q_{W^{1,p}(\Omega\setminus\Gamma)}.\end{aligned} \tag{3.9}$$

Now combining (3.7)–(3.9), we obtain the claimed result. □

REMARK 3.7 Note that the proof does not depend on d being equal to two. The estimate is thus also proved for the three-dimensional case.

4. Cracked domains

In the case of an interface Γ that ends within the domain we have yet to define the interpolant for those elements that contain endpoints of the interface. Consider the triangle ABC as shown in Fig. 3. The interface ends at point T, but point Q can still be defined by suitably continuating the segment PT, and the subdivision of the element is identical to the case of a fully-cut triangle. The simplest possibility is to adopt the same basis functions as in the fully cut case, in which case Theorem 3.2 applies without any modification to the proof.

However, if the basis functions of the fully cut element are used, then property P4 does not hold. The interpolant $\mathcal{I}_h u$ is discontinuous not only at Γ but also along the edge AC, since φ_A and φ_C are different as seen from the two elements that share the edge AC. For some formulations, such as those that make use of the pressure Poisson equation, it is usual to have $W_h \in H^1(\Omega\setminus\Gamma)$. This will not hold with the interpolant proposed above because of the discontinuities at the partially cut elements just discussed. A special treatment for the partially cut elements is thus needed.

As explained by Ausas *et al.* (2010), continuity everywhere outside Γ can easily be recovered in both two and three dimensions without deteriorating the accuracy of the interpolation. The basis functions needed for this fix, which are piecewise $\mathbb{P}_1$ inside each of the subtriangles APQ, BCP and CQP as

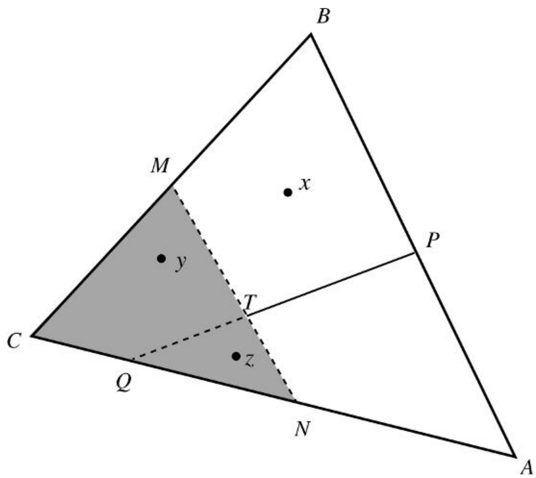

FIG. 3. Scheme of an element (the triangle ABC) containing an endpoint T of the interface, which is represented by the segment PT.

before, are defined to be continuous along the edge AC as follows. Let

$$a = \frac{|CQ|}{|AC|},$$

and then we have

$$\varphi_A(A) = 1, \quad \varphi_B(A) = 0, \quad \varphi_C(A) = 0, \tag{4.1}$$
$$\varphi_A(B) = 0, \quad \varphi_C(B) = 0, \tag{4.2}$$
$$\varphi_A(C) = 0, \quad \varphi_B(C) = 0, \quad \varphi_C(C) = 1z, \tag{4.3}$$
$$\varphi_A(P^+) = 1, \quad \varphi_B(P^+) = 0, \varphi_C(P^+) = 0, \tag{4.4}$$
$$\varphi_A(P^-) = 0, \quad \varphi_B(P^-) = 1, \varphi_C(P^-) = 0, \tag{4.5}$$
$$\varphi_A(Q) = a, \quad \varphi_B(Q) = 0, \varphi_C(Q) = 1 - a. \tag{4.6}$$

The corresponding functions are plotted in Fig. 4. Note that φ_C is simply the standard $\mathbb{P}_1$ basis function since the interface does not intersect any edge containing the point C.

It is easily checked that this basis satisfies properties P1–P4 above. A three-dimensional version, also satisfying the same properties, has also been introduced (Ausas *et al.*, 2010). The rest of this section is devoted to extending the interpolation estimate to this case. There exist now the following three subsets of elements: $\mathcal{R}_h$ contains those elements *not intersected* by Γ, $\mathcal{S}_h$ contains those elements *fully cut* by Γ, and $\mathcal{Z}_h$ contains those elements *partially cut* by Γ in which the new basis has been adopted.

This last subset is the only novelty with respect to the proof of Lemma 3.6. In particular, from the quasiuniformity of the mesh and the regularity of Γ, we assume that the number of elements in $\mathcal{Z}_h$ is $N_{\mathcal{Z}_h} \leqslant c_z h^{-d+2}$. In particular, in two dimensions, the number of endpoints of Γ is assumed to be finite.

Our purpose is to prove that Theorem 3.2 still holds if Γ ends within the domain and the basis defined by (4.1)–(4.6) is adopted in the partially cut elements.

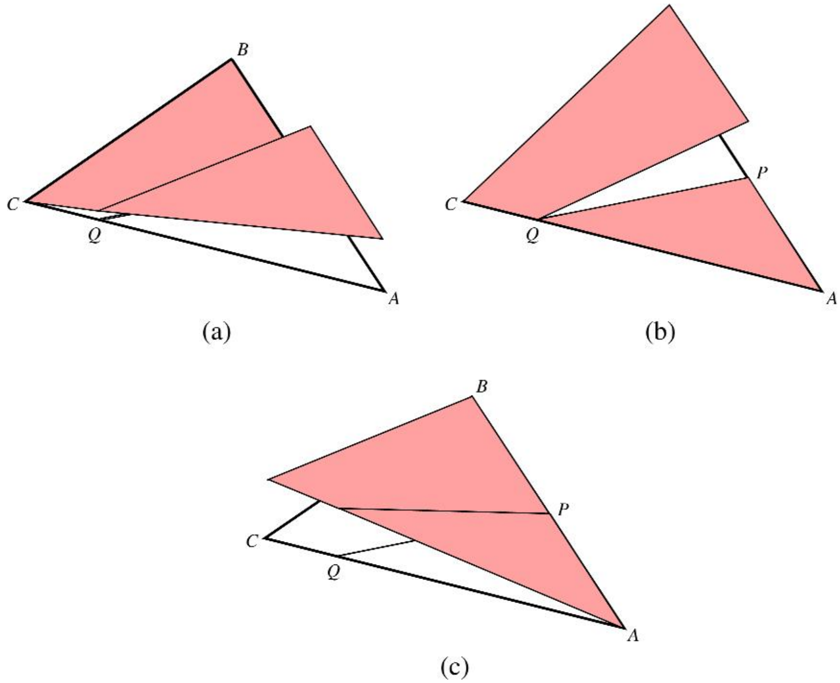

FIG. 4. Basis functions for a ‘cracked’ element: (a) φ_A, (b) φ_B and (c) φ_C.

We begin, as in the proof of Lemma 3.6, by decomposing as follows:

$$\|u-\mathcal{I}_h u\|_{L^q(\Omega)} \leqslant \left(\sum_{K\in\mathcal{R}_h} \|u-\mathcal{I}_K u\|^q_{L^q(K)}\right)^{\frac{1}{q}} + \left(\sum_{K\in\mathcal{S}_h} \|u-\mathcal{I}_K u\|^q_{L^q(K)}\right)^{\frac{1}{q}} + \left(\sum_{K\in\mathcal{Z}_h} \|u-\mathcal{I}_K u\|^q_{L^q(K)}\right)^{\frac{1}{q}}. \tag{4.7}$$

Since the bounds for the two first terms have already been established, it only remains to show that there exists a constant C such that

$$\left(\sum_{K\in\mathcal{Z}_h} \|u-\mathcal{I}_K u\|^q_{L^q(K)}\right)^{\frac{1}{q}} \leqslant Ch^{1+\frac{1}{q}-\frac{1}{p}}|u|_{W^{1,p}(\Omega\setminus\Gamma)}. \tag{4.8}$$

The difficulty lies in that Lemma 3.5 does not hold for this new interpolant because it is based on applying Lemma 3.4 to the open set

$$\{x\in K\setminus\Gamma | \varphi_A(x)\neq 0\},$$

which, with the new basis, is not star-shaped with respect to A. As depicted in Fig. 3, points x exist with $\varphi_A(x)\neq 0$ and such that the straight segment $\overline{xA}$ intersects Γ, making the argument of Lemma 3.5

invalid. We explain here how to tackle this difficulty in the case of a straight interface (PT is a straight segment). Its extension to the curved case is not difficult, but requires some technicalities.

The idea is to construct a path that avoids crossing Γ. For this purpose, let us denote by Δ_1 the triangle APQ, by Δ_2 the quadrilateral $BCQP$, and let us define a third convex set Δ_3 (in Fig. 3 it corresponds to the triangle CNM) that overlaps with the other two. Further, we define $\omega_1 = \Delta_1 \cap \Delta_3$ and $\omega_2 = \Delta_2 \cap \Delta_3$. Our path from the point x to A will consist of three segments, namely, $\overline{xy}$, with $y \in \omega_2$, then $\overline{yz}$, with $z \in \omega_1$, and finally $\overline{zA}$. With these definitions, Lemma 3.6 is adapted to the partially cut element as follows.

LEMMA 4.1 Let K be a partially cut element and let A be a vertex. Then, for $p > d$ and $1 \leqslant q \leqslant p$ we have

$$\|w - w(A)\|_{L^q(K)} \leqslant \frac{2p}{p-d}\left[1 + \left(\frac{|\Delta_2|}{|\omega_1|}\right)^{\frac{1}{q}}\right] \mathrm{diam}(K)|K|^{\frac{1}{q}-\frac{1}{p}}|w|_{W^{1,p}(K\setminus\Gamma)}. \tag{4.9}$$

Proof. We begin by decomposing as follows:

$$\|w - w(A)\|_{L^q(K)} \leqslant \|w - w(A)\|_{L^q(\Delta_1)} + \|w - w(A)\|_{L^q(\Delta_2)}. \tag{4.10}$$

Since Δ_1 is convex and $A \in \overline{\Delta_1}$, Lemma 3.4 immediately provides a suitable bound for the first term, that is,

$$\|w - w(A)\|_{L^q(\Delta_1)} \leqslant \frac{p}{p-d}\mathrm{diam}(\Delta_1)|\Delta_1|^{\frac{1}{q}-\frac{1}{p}}\, |w|_{W^{1,p}(\Delta_1)}. \tag{4.11}$$

We thus now turn to the second term

$$\|w - w(A)\|^q_{L^q(\Delta_2)} = \int_{\Delta_2} |w(x) - w(A)|^q \,\mathrm{d}x = \frac{1}{|\omega_1||\omega_2|}\int_{\omega_1}\int_{\omega_2}\int_{\Delta_2} |w(x) - w(y) + w(y) - w(z) + w(z) - w(A)|^q \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Defining $\mathcal{B} = \Delta_2 \times \omega_2 \times \omega_1$ and with $F_1(x, y, z) = w(x) - w(y)$, $F_2(x, y, z) = w(y) - w(z)$ and $F_3(x, y, z) = w(z) - w(A)$, we arrive at

$$\begin{aligned}\|w - w(A)\|_{L^q(\Delta_2)} &= \frac{1}{(|\omega_1||\omega_2|)^{\frac{1}{q}}}\|F_1 + F_2 + F_3\|_{L^q(\mathcal{B})} \\ &\leqslant \frac{1}{(|\omega_1|\,|\omega_2|)^{\frac{1}{q}}}(\|F_1\|_{L^q(\mathcal{B})} + \|F_2\|_{L^q(\mathcal{B})} + \|F_3\|_{L^q(\mathcal{B})}).\end{aligned} \tag{4.12}$$

Now we establish bounds for each term. In particular, since both x and y belong to Δ_2, we have from Lemma 3.4 that

$$\begin{aligned}\|F_1\|^q_{L^q(\mathcal{B})} &= \int_{\omega_1}\int_{\omega_2}\int_{\Delta_2} |w(x) - w(y)|^q \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\leqslant |\omega_1||\omega_2|c_p^q \mathrm{diam}(\Delta_2)^q|\Delta_2|^{1-\frac{q}{p}}|w|^q_{W^{1,p}(\Delta_2)},\end{aligned} \tag{4.13}$$

where we have defined $c_p = p/(p-d)$. Similarly,

$$\|F_2\|^q_{L^q(\mathcal{B})} = \int_{\omega_1}\int_{\omega_2}\int_{\Delta_2} |w(y) - w(z)|^q \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leqslant |\Delta_2||\omega_2| c_p^q \mathrm{diam}(\Delta_3)^q |\Delta_3|^{1-\frac{q}{p}} \, |w|^q_{W^{1,p}(\Delta_3)} \tag{4.14}$$

and

$$\|F_3\|^q_{L^q(\mathcal{B})} = \int_{\omega_1}\int_{\omega_2}\int_{\Delta_2} |w(z) - w(A)|^q \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leqslant |\Delta_2||\omega_2| c_p^q \mathrm{diam}(\Delta_1)^q \, |\Delta_1|^{1-\frac{q}{p}} |w|^q_{W^{1,p}(\Delta_1)}. \tag{4.15}$$

Inserting (4.12)–(4.15) and (4.11) into (4.10), the claim is proved. □

To complete the proof of (4.8) we proceed as in Lemmata 3.5 and 3.6, adding up the contributions of all elements in $\mathcal{Z}_h$, to get

$$\left(\sum_{K\in\mathcal{Z}_h} \|u - \mathcal{I}_K u\|^q_{L^q(K)}\right)^{\frac{1}{q}} \leqslant \mathcal{A}(h) h^{1+\frac{1}{q}-\frac{1}{p}} |u|_{W^{1,p}(\Omega\setminus\Gamma)}, \tag{4.16}$$

where

$$\mathcal{A}(h) = \frac{2c_z p(d+1)}{p-d} \max_{K\in\mathcal{Z}_h}\left(1 + \frac{|\Delta_2(K)|^{\frac{1}{q}}}{|\omega_1(K)|^{\frac{1}{q}}}\right) h^{\frac{1}{q}-\frac{1}{p}}, \tag{4.17}$$

which implies (4.8) under the *technical* assumption that the family of triangulations $\mathcal{T}_h$ is such that there exists a constant C such that

$$\mathcal{A}(h) \leqslant C \quad \forall\, h > 0.$$

This technical assumption in practice requires that the interface does not end exactly at an edge (a face in three dimensions), and we have never observed any pathological behaviour in the interpolation when the endpoint lies very close to an edge. Note that $h^{\frac{1}{q}-\frac{1}{p}}$ tends to 0 with h, so that the ratio $|\Delta_2(K)|/|\omega_1(K)|$ may even diverge without harming the convergence order if the divergence is mild enough.

5. Numerical experiments

As a complement to the numerical experiments shown in Ausas *et al.* (2010), let us consider the interpolation of the function

$$u(r,\theta) = r^2\,\mathrm{e}^{-4r}\sin\left(\frac{\theta}{4}\right), \tag{5.1}$$

where r is the distance from a point z chosen randomly in $(-0.25, 0.25)\times(-0.25, 0.25)$ and θ is the angle measured from some randomly chosen $\theta_0 \in [0, 2\pi)$. The domain Ω is taken as $(-2,2)\times(-2,2)$. An example of the interpolated function for an unstructured mesh with $h = 0.1$, $z = (0.1, 0.2)$ and $\theta_0 = \frac{\pi}{3}$ is shown in Fig. 5. Note how the function becomes rough near the discontinuity line Γ because the interpolant is constant along the edges on each side of Γ.

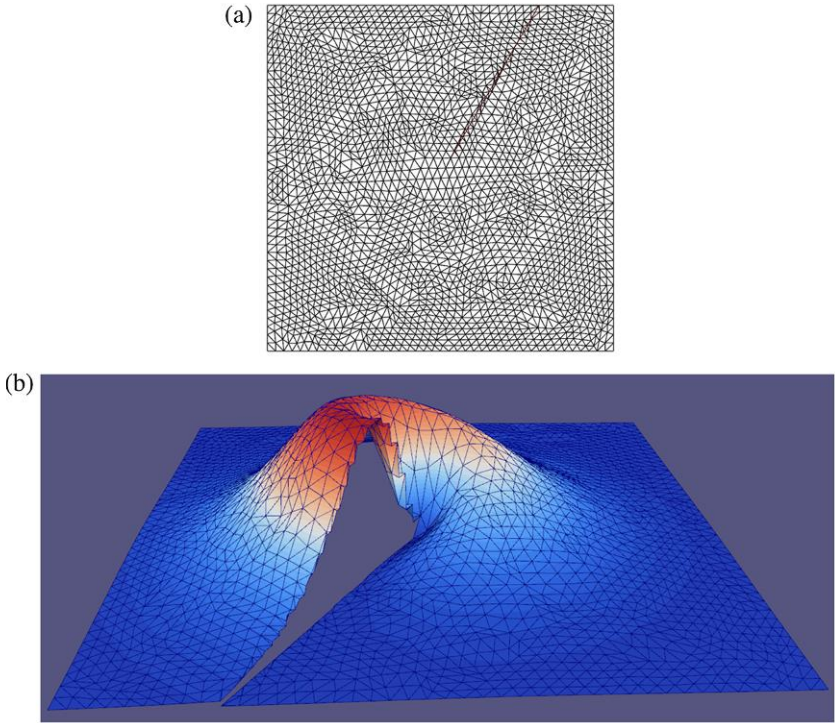

FIG. 5. (a) Unstructured mesh with $h = 0.1$ and (b) the discontinuous function u interpolated on it with the proposed interpolant.

We investigate here two issues. The first is the robustness of the interpolation with respect to the exact position of Γ in the mesh. For this purpose, we randomly generate 10,000 functions by varying z and θ_0. The distribution of the interpolation error

$$e_0 = \|u - \mathcal{I}_h u\|_{L^2(\Omega)} \tag{5.2}$$

is shown in Fig. 6 for the mesh shown in Fig. 5 ($h = 0.1$) and a refined mesh obtained by dividing each triangle into four ($h = 0.05$). The mean $L^2(\Omega)$-errors for each mesh are 4.74×10^{-4} and 1.46×10^{-4}. This corresponds to a behavior of the mean of the error as $h^{1.7}$, consistent with the one predicted by the theoretical estimate of the previous sections ($\mathrm{O}(h^{1.5})$).

The ratios of the maximum to minimum errors are observed to be rather small, namely, 3.44 for the first mesh and 2.80 for the second. The interpolation accuracy depends, of course, on the way the triangles are cut, but no configuration leads to a 'disastrous' interpolant.

The second issue investigated here is the approximation properties in $H^1(\Omega)$. For this purpose, we choose the same function as above, with $z = (0.1, 0.2)$ and $\theta_0 = \frac{\pi}{3}$, and perform a mesh refinement study measuring

$$e_1 = \|\nabla u - \nabla(\mathcal{I}_h u)\|_{L^2(\tilde{\Omega})}, \tag{5.3}$$

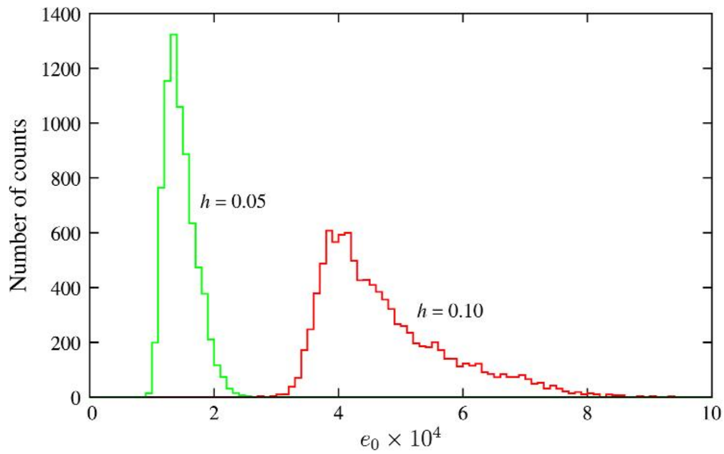

FIG. 6. Histograms of the distributions of the interpolation $L^2(\Omega)$-errors, for two meshes, after 10,000 random realizations.

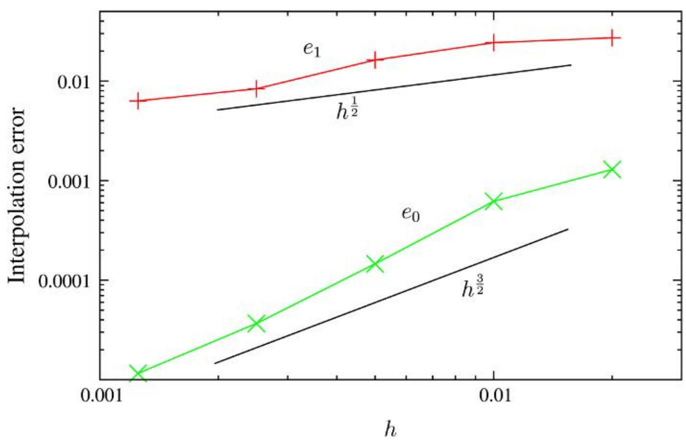

FIG. 7. Plots of the interpolation errors e_0 and e_1 as functions of h.

where $\tilde{\Omega}$ is the subset of Ω and where $\mathcal{I}_h u$ is continuous. The results are shown in Fig. 7 and show evidence of an interpolation order of $h^{\frac{1}{2}}$, which is quite logical since, in the interior of the cut elements, ∇u is not approximated at all.

6. Concluding remarks

A new finite-element space W_h has been analysed that has the same unknowns as the $\mathbb{P}_1$-conforming space but consists of functions that are discontinuous across a given interface Γ, which is assumed to not be aligned with the mesh.

The interpolation estimate yields a convergence rate in $L^2(\Omega)$ of order $h^{\frac{3}{2}}$ for functions that are smooth outside Γ. This rate, which is sharp as shown by numerical experiments in Ausas *et al.* (2010) and Section 5, is a significant improvement with respect to the accuracy of continuous spaces $\left(\text{of } O\left(h^{\frac{1}{2}}\right)\right)$ but is still suboptimal. However, such a convergence rate implies that the space W_h, when taken as a pressure space, will *not* limit the accuracy of a (Navier–)Stokes calculation neither in equal-order velocity–pressure approximations nor in the minielement approximation. In fact, in both cases the global accuracy is limited by the $H^1(\Omega)$-accuracy of the velocity space, which is at most $O(h)$.

In the provided estimates the interface Γ is assumed to be exact. In finite-element applications of the space, however, the exact interface is some $\hat{\Gamma}$, and Γ is a suitable approximation thereof that renders the integrals computable. Let us assume that both $\hat{\Gamma}$ and Γ are sufficiently smooth and that the distance between them satisfies

$$\delta := \mathrm{dist}(\hat{\Gamma}, \Gamma) \leqslant Ch^r. \tag{6.1}$$

For example, if Γ is piecewise affine, then we expect that $r = 2$. Given a function $\hat{u}$ that is discontinuous at $\hat{\Gamma}$ and belongs to $W^{1,p}(\Omega \setminus \hat{\Gamma})$, it must be approximated by some function $u \in W^{1,p}(\Omega \setminus \Gamma)$ before applying the interpolation estimate of Theorem 3.2. This introduces an additional error that, under suitable assumptions, is of the order

$$\|\hat{u} - u\|_{L^q(\Omega)} \leqslant C\|\hat{u}\|_{W^{1,q}(\Omega\setminus\hat{\Gamma})}\delta^{\frac{1}{q}} \leqslant Ch^{\frac{r}{q}}.$$

Note that, for piecewise affine Γ, this error, in the L^2-norm, is $O(h)$. For the interpolation estimate $O\left(h^{\frac{3}{2}}\right)$, obtained for the exact interface case, to remain true in the case of an approximate interface, it must be guaranteed that $r \geqslant 3$ in (6.1), which can be achieved, for example, with a piecewise parabolic Γ.

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