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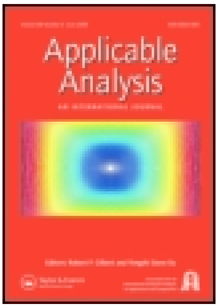

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A posteriori estimate and asymptotic partial domain decomposition

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A posteriori estimate and asymptotic partial domain decomposition

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The method of asymptotic partial decomposition of a domain aims at replacing a 3D or 2D problem by a hybrid problem 3D − 1D; or 2D − 1D, where the dimension of the problem decreases in part of the domain. The location of the junction between the heterogeneous problems is asymptotically estimated in certain circumstances, but for numerical simulations it is important to be able to determine the location of the junction accurately. In this article, by reformulating the problem in a mixed formulation context and by using an *a posteriori* error estimate, we propose an indicator of the error due to a wrong position of the junction. Minimizing this indicator allows us to determine accurately the location of the junction. Some numerical results are presented for a toy problem.

Keywords: asymptotic partial domain decomposition; *a posteriori* error estimates; error indicator

AMS Subject Classifications: 35F40; 65

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1. Introduction

The method of asymptotic partial decomposition of a domain (MAPDD) originates from the works of Panasenko [1]. The idea is to replace an original 3D or 2D problem by a hybrid one 3D − 1D; or 2D − 1D, where the dimension of the problem decreases in part of the domain. Effective solution methods for the resulting hybrid problem have recently become available for several systems (linear/nonlinear, fluid/solid, etc.) which allow for each subproblem to be computed with an independent black-box code [2–4]. The location of the junction between the heterogeneous problems is asymptotically estimated in the works of Panasenko [5]. MAPDD has been designed for handling problems where a small parameter appears, and provides a series expansion of the solution with solutions of simplified problems with respect

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to this small parameter. In the problem considered here, no small parameter exists, but due to geometrical considerations concerning the domain Ω it is assumed that the solution does not differ very much from a function which depends only on one variable in a part of the domain. The MAPDD theory is not suited for such a context, but if this theory is applied formally it does not provide any error estimate. The *a posteriori* error estimate proved in this article, is able to measure the discrepancy between the exact solution and the hybrid solution which corresponds to the zero-order term in the series expansion with respect to a small parameter when it exists.

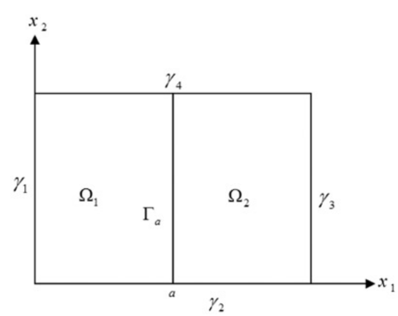
Numerically, independently of the existence of an asymptotical estimate of the location of the junction, it is essential to detect with accuracy the location of the junction. Let us also mention the interest of locating with accuracy the position of the junction in blood flows simulations [6]. Here the method proposed is to determine the location of the junction (i.e. the location of the boundary Γ in the example treated) by using optimization techniques. First it is shown that MAPDD can be expressed with a mixed domain decomposition formulation (as in [5,7–12]) in two different ways. Then it is proposed to use an *a posteriori* error estimate for locating the best position of the junction. A posteriori error estimates have been extensively used in optimization problems, the reader is referred to, e.g. [8,9].

In the following, the toy problem for which the method is presented is given, and the introduction section is ended with the mixed formulation of the domain decomposition of the problem. Section 2 is dedicated to the two asymptotic decompositions proposed for a given location of the interface Γ. One asymptotic decomposition is based on a particular mortar subspace (the constant functions on Γ), and the other one is based on coupling a partial differential equation (PDE) with an ordinary differential equation (ODE). In Section 3, *a posteriori* error estimates are given and an indicator is proposed. In Section 4, the optimal location of the junction is found by minimizing the indicator. Numerical results are provided showing the efficiency of the proposed method.

Let f be a regular function defined by

$$f(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & 0 < x_1 < a, \\ f_2(x_1), & a < x < 1. \end{cases} \tag{1}$$

The domain $\Omega = (0, 1) \times (0, 1)$ is decomposed in two subdomains $\Omega_1 = (0, a) \times (0, 1)$ and $\Omega_2 = (a, 1) \times (0, 1)$, the boundary $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, and the boundary $\partial\Omega$ are divided into four subparts $\gamma_1 = \{0\} \times (0, 1)$ $\gamma_2 = (0, 1) \times \{0\}$ $\gamma_3 = \{1\} \times (0, 1)$ $\gamma_4 = (0, 1) \times \{1\}$.



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Let $U \in H^2(\Omega)$ be the solution to:

$$\begin{cases} -\Delta U(x_1, x_2) = f(x_1, x_2), & \text{in } \Omega, \\ \partial_n U = 0, & \text{on } \gamma_{2i};\ 1 \le i \le 2; \\ U = 0, & \text{on } \gamma_{2i-1};\ 1 \le i \le 2. \end{cases} \tag{2}$$

Now let us give a formulation of the problem in the domain decomposition context with a L^2-mortar subspace. We define the following functional spaces:

$$\begin{aligned} &{}_0H^1(\Omega_1) = \{\varphi \in H^1(\Omega_1),\ \varphi|_{\gamma_1} = 0\}, \\ &{}_0H^1(\Omega_2) = \{\varphi \in H^1(\Omega_2),\ \varphi|_{\gamma_3} = 0\}, \\ &V =_0 H^1(\Omega_1) \times_0 H^1(\Omega_2), \\ &W =_0 H^1(\Omega_1) \times_0 H^1(\Omega_2) \cap \{D_2\varphi|_{\Omega_2} = 0\}, \\ &\Lambda = L^2(\Gamma), \end{aligned} \tag{3}$$

equipped with the norms

$$|v|_1^2 = \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_i \cdot \nabla v_i \, \mathrm{d}x_1 \, \mathrm{d}x_2, \qquad \|\xi\|_\Lambda^2 = \int_\Gamma \xi^2 \, \mathrm{d}x_2. \tag{4}$$

Let us define $(u_1, u_2, \lambda) \in V \times \Lambda$ solution to

$$\begin{cases} \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, \mathrm{d}x_1 \, \mathrm{d}x_2 + \int_\Gamma \lambda(v_1 - v_2) \mathrm{d}x_2 = \sum_{i=1}^{2} \int_{\Omega_i} f v_i \, \mathrm{d}x_1 \, \mathrm{d}x_2, & \forall v \in V \\ \displaystyle\int_\Gamma \xi(u_1 - u_2) \, \mathrm{d}x_2 = 0, \quad \forall \xi \in \Lambda. \end{cases} \tag{5}$$

Introduce the following bilinear forms:

$$\begin{aligned} &a : V \times V \longrightarrow \mathbb{R} \\ &\qquad u, v \longmapsto a(u, v) = \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \nabla v_i \, \mathrm{d}x_1 \mathrm{d}x_2, \\ &b : \Lambda \times V \longrightarrow \mathbb{R} \\ &\qquad \xi, v \longmapsto b(\xi, v) = \int_\Gamma \xi(v_1 - v_2) \mathrm{d}x_2. \end{aligned}$$

We have the following result.

Lemma 1.1 *Assume $f \in L^2(\Omega)$ then, there exists a unique $(u_1, u_2, \lambda) \in V \times \Lambda$ solution to problem* (5). *Moreover, we have $u_i = U|_{\Omega_i}$ for $1 \le i \le 2$.*

Proof Let K be the closed subset of space V defined by

$$K = \{v \in V;\ v_1 - v_2|_\Gamma \in \Lambda^\perp\}.$$

The existence result is a consequence of the following inf-sup condition: there exists $0 < \beta$ such that

$$\inf_{(w,\mu) \in K \times \Lambda} \ \sup_{(v,\xi) \neq (0,0) \in V \times \Lambda} \frac{a(w, v) + b(\mu, v) + b(\xi, w)}{\|\xi\|_\Lambda + |v|_1} \ge \beta. \tag{6}$$

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To see this, take $(w, \mu) \in K \times \Lambda$ with $|w|_1 + \|\mu\|_\Lambda = 1$ and choose $v = w + w_1$ with w_1 solution to

$$\begin{cases} -\Delta w_1(x_1, x_2) = 0, \text{ in } \Omega_1, \\ \partial_n w_1 = \mu, \text{ on } \Gamma \text{ and } w_1 = 0, \text{ on } \partial\Omega \setminus \Gamma. \end{cases} \tag{7}$$

Let us decompose the space $V : K \bigoplus K^\perp = V$ where the orthogonality is defined with the inner product induced by $a(.,\, .)$. We have $(w_1, 0) \in K^\perp$ since assuming $(w_1, 0) \in K$ leads to $w_1|_\Gamma = 0$ which combined with the first equation of (7) would lead to $w_1 = 0$.

For all $v_1 \in H^1(\Omega_1)$, $v_1 \,|_{\partial\Omega_1 \setminus \Gamma} = 0$

$$\int_{\Omega_1} \nabla w_1 \nabla v_1 \mathrm{d}x = \int_\Gamma \mu v_1 \mathrm{d}x_2$$

and the following estimate holds true:

$$|w_1|_1 = \sup_{\substack{v_1 \in H^1(\Omega_1) \\ v_1|_{\partial\Omega_1 \setminus \Gamma} = 0}} \int_{\Omega_1} \nabla w_1 \nabla v_1 \mathrm{d}x = \sup_{\substack{v_1 \in H^1(\Omega_1) \\ v_1|_{\partial\Omega_1 \setminus \Gamma} = 0}} \int_\Gamma \mu v_1 \mathrm{d}x_2,$$

$$|w_1|_1 \le c_2 \|\mu\|_{0,\Gamma}. \tag{8}$$

Since the space $\tilde{H}^{\frac{1}{2}}(\Gamma) = \{\varphi \in H^{\frac{1}{2}}(\Gamma)$ the extension of which by 0 belongs to $H^{\frac{1}{2}}(\partial\Omega_1)\}$ is densely embedded in $L^2(\Gamma)$, there exists $\mu_\varepsilon \in \tilde{H}^{\frac{1}{2}}(\Gamma)$: verifying:

$$\|\mu - \mu_\varepsilon\|_{0,\Gamma} \le \varepsilon.$$

We choose ε such that:

$$|w_1|_1 = \sup_{\substack{v_1 \in H^1(\Omega_1) \\ v_1|_{\partial\Omega_1 \setminus \Gamma} = 0}} \int_\Gamma \mu_\varepsilon v_1 + v_1(\mu - \mu_\varepsilon) \mathrm{d}x_2 \ge (\|\mu\|_0 - \varepsilon)^2 - (\|\mu\|_0 + \varepsilon)\varepsilon \ge \frac{1}{2}\|\mu\|_0^2.$$

So the following quantity

$$I = \sup_{(v,\xi) \neq (0,0) \in V \times \Lambda} \frac{a(w, v) + b(\mu, v) + b(\xi, w)}{\|\xi\|_\Lambda + |v|_1}$$

with $\xi = \mu$ and estimate (8) verifies the following estimate

$$I \ge \frac{1}{2} \frac{|w|_1^2 + \|\mu\|_0^2}{\max(1, c_2)} \ge \frac{1}{2\max(1, c_2)} \inf_{\substack{x+y=1 \\ x \ge 0,\, y \ge 0}} [x^2 + y^2] \ge \frac{1}{4\max(1, c_2)} = \beta.$$

Now we integrate by parts in the bilinear form $a(\cdot,\, \cdot)$. Choosing $v_i \in \mathcal{D}(\Omega_i)$, we deduce $\nabla(u_i - U)|_{\Omega_i} = 0$. Thanks to Dirichlet's conditions we have: $u_i - U|_{\Omega_i} = 0$.

Choose $v_i \in \mathcal{D}(\overline{\Omega}_i)$; $v_i|_{\gamma_{2i}} = 0$; $1 \le i \le 2$. Integrating by parts in the bilinear form $a(\cdot,\, \cdot)$ we have:

$$\int_\Gamma \partial_{n_1} u_1 v_1 + \partial_{n_2} u_2 v_2 + \lambda(v_1 - v_2)\, \mathrm{d}x_2 = 0. \tag{9}$$

Taking $v_1 = 0$ we have $\partial_{n_2} u_2 = \lambda$ in $L^2(\Gamma)$ and for $v_2 = 0$ we have $\partial_{n_1} u_1 = -\lambda$ in $L^2(\Gamma)$. The conditions are expressed in L^2 since, $u_i \in H^2(\Omega_i)$, $1 \le i \le 2$.

Since $b(\xi, u) = 0$ for all $\xi \in L^2(\Gamma)$, we deduce that $u_1 = u_2$ on Γ. ■

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2. Asymptotic domain decomposition

In this section, we propose two approximate domain decomposition problems by using different mortar subspaces or different spaces for the solution. Let $\Lambda_0 = \mathrm{span}\{1\}$ and let us define $(\tilde{u}_1, \tilde{u}_2, \lambda_0) \in V \times \Lambda_0$ solution of

$$\begin{cases} a(\tilde{u}, v) + b(\lambda_0, v) = \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} f v_i \,\mathrm{d}x_1 \mathrm{d}x_2, \quad \forall v \in V, \\ b(\xi, \tilde{u}) = 0, \quad \forall \xi \in \Lambda_0. \end{cases} \tag{10}$$

LEMMA 2.1 *Assume $f \in L^2(\Omega)$, then, there exists a unique $(\tilde{u}_1, \tilde{u}_2, \lambda_0) \in V \times \Lambda_0$ solution to problem* (10). *Moreover, we have*

$$\partial_{n_1} \tilde{u}_1 = -\partial_{n_2} \tilde{u}_2, \text{ in } L^2(\Gamma), \quad \tilde{u}_2|_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma \tilde{u}_1 \,\mathrm{d}x_2.$$

Proof The existence result is a consequence of the inf-sup condition, which is proved in the same way as in Lemma 1.1 with $w_1 = cx_1$, and

$$\Lambda_0^\perp = \{\varphi \in L^2(\Gamma); \int_\Gamma \varphi(x_2) \mathrm{d}x_2 = 0\} \Rightarrow (w_1, 0) \in K^\perp.$$

Integrating by parts in (10), thus, since $\tilde{u}_i \in H^2(\Omega_i)$, we have

$$\partial_{n_2} \tilde{u}_2|_\Gamma = \lambda_0 \in \Lambda_0.$$

Take $v_2 = 0$, whatever v_1 is:

$$\int_\Gamma (\partial_{n_1} \tilde{u}_1 + \lambda_0) v_1 \mathrm{d}x_2 = 0 \Rightarrow \partial_{n_1} \tilde{u}_1 + \lambda_0 = 0 \text{ in } \tilde{H}^{\frac{1}{2}}(\Gamma)' \Rightarrow \partial_{n_1} \tilde{u}_1 = -\lambda_0.$$

Now, let us prove that $\tilde{u} \in W$. Since λ_0 is constant, it is easy to prove that $\tilde{u}_2(x_1)$ solution to

$$\begin{cases} -\tilde{u}_2''(x_1) = f_2(x_1), \text{ in } a < x_1 < 1, \\ \tilde{u}_2'(x_1) = \lambda_0, \quad \tilde{u}_2(1) = 0, \end{cases}$$

is the unique solution $\tilde{u}_2$ in the domain Ω_2.

The condition $b(1, \tilde{u}) = 0$ implies $\tilde{u}_2 = \frac{1}{\Gamma} \int_\Gamma \tilde{u}_1 \,\mathrm{d}x_2$. ■

Remark 1 The previous result provides a justification for coupling an ODE with a PDE in some circumstances which can also be justified with asymptotic developments.

Now, set $\Lambda_2 = L^2(\Gamma)$ as mortar subspace, and let us define $(\hat{u}_1, \hat{u}_2, \lambda_2) \in W \times \Lambda_2$ solution to

$$\begin{cases} a(\hat{u}, v) + b(\lambda_2, v) = \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} f v_i \,\mathrm{d}x_1 \mathrm{d}x_2, \quad \forall v \in V, \\ b(\xi, \hat{u}) = 0, \quad \forall \xi \in \Lambda_2. \end{cases} \tag{11}$$

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LEMMA 2.2 *Assume* $f \in L^2(\Omega)$, *then, there exists a unique* $(\hat{u}_1, \hat{u}_2, \lambda_2) \in W \times \Lambda_2$ *solution to problem* (11). *Moreover, we have*

$$\partial_{n_2}\hat{u}_2 = -\frac{1}{|\Gamma|}\int_\Gamma \partial_{n_1}\hat{u}_1 \,\mathrm{d}x_2 \quad \hat{u}_1 = \hat{u}_2, \text{ in } L^2(\Gamma).$$

Proof The space W is a closed subspace of V thus the existence is proved in the same way as in the previous lemma. The identity (9) with $v_1 = 0$ reads: for every $v_2 \in L^2(\Gamma)$

$$\int_\Gamma (\partial_{n_2}\hat{u}_2 - \lambda_2)v_2 \mathrm{d}x_2 = 0. \tag{12}$$

Since $\partial_{n_2}\hat{u}_2 - \lambda_2 \in \Lambda_2^\perp$ we conclude that $\partial_{n_2}\hat{u}_2 = \lambda_2$. Take $v_2 = 0$, for every $v_1 \in L^2(\Gamma)$, identity (9) reads:

$$\int_\Gamma (\partial_{n_1}\hat{u}_1 + \lambda_2)v_1 \mathrm{d}x_2 = 0 \Rightarrow \partial_{n_1}\hat{u}_1 = -\lambda_2, \text{ in } \Lambda_2. \tag{13}$$

Since $\hat{u}_2 \in W$, the relation (12) reads: $\partial_{n_2}\hat{u}_2 = -\frac{1}{|\Gamma|}\int_\Gamma \partial_{n_1}\hat{u}_1 \,\mathrm{d}x_2$. The condition $b(\xi, \hat{u}) = 0$ for every $\xi \in \Lambda_2$ implies $\hat{u}_2 = \hat{u}_1$. ■

3. *A posteriori* error estimates

In this section an *a posteriori* error estimate is derived for the error between the exact solution of the domain decomposition formulation of the problem, and the approximate solution by using a mortar subspace.

Let us define the operator

$T\colon V \times Ł^2(\Gamma) \to V' \times Ł^2(\Gamma)'$ by

$$\langle T(\tilde{u}, \lambda_0), (v, \xi)\rangle = a(\tilde{u}, v) + b(\lambda_0, v) + b(\xi, \tilde{u}).$$

Define the error e by

$$e = (u - \tilde{u}, \lambda - \lambda_0). \tag{14}$$

In what follows, an indicator for the error is proposed.

The error equation reads:

$$\begin{aligned}\langle Te, (v, \xi)\rangle &= a((u - \tilde{u}), v) + b(\lambda - \lambda_0, v) + b(\xi, u - \tilde{u}) \\ &= -\int_\Gamma (\partial_{n_1}\tilde{u}_1 v_1 + \partial_{n_2}\tilde{u}_2 v_2 \mathrm{d}x_2 - \int_\Gamma \lambda_0(v_1 - v_2)\mathrm{d}x_2 - \int_\Gamma \xi(\tilde{u}_1 - \tilde{u}_2)\,\mathrm{d}x_2 \\ &= \mathcal{L}_{\tilde{u},\lambda_0}(v, \xi).\end{aligned} \tag{15}$$

LEMMA 3.1 *Assume* $f \in L^2(\Omega)$, *then, the following estimate holds true.*

$$\frac{\|\mathcal{L}_{\tilde{u},\lambda_0}\|_*}{\|T\|_{\mathcal{L}}} \le \|e\| \le \|\mathcal{L}_{\tilde{u},\lambda_0}\|_* \|T^{-1}\|_{\mathcal{L}}, \tag{16}$$

with

$$\|\mathcal{L}_{\tilde{u},\lambda_0}\|_* = \|\tilde{u}_1 - \frac{1}{|\Gamma|}\int_\Gamma \tilde{u}_1 \,\mathrm{d}x_2\|_{0,\Gamma},$$
$$\beta \le \|T\|_{\mathcal{L}}; \quad \|T^{-1}\|_{\mathcal{L}} \le \frac{1}{\beta}. \tag{17}$$

Proof We have to evaluate:

$$\|\mathcal{L}_{\tilde{u},\lambda_0}\|_* = \sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{a(u-\tilde{u},v) + b(\lambda-\lambda_0,v) + b(\xi,u-\tilde{u})}{\|\xi\|_{0,\Gamma} + |v|_1}. \tag{18}$$

Observe that by integrating by parts in the bilinear form $a(\cdot,\cdot)$, we have

$$a(u-\tilde{u},v) = -\int_\Gamma \partial_{n_1}\tilde{u}_1 v_1 + \partial_{n_2}\tilde{u}_2 v_2 dx_2 - \int_\Gamma \lambda(v_1 - v_2)\mathrm{d}x_2.$$

Gathering (18) with the previous inf-sup condition, we deduce

$$\|e\| \le \frac{1}{\beta} \sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{-\int_\Gamma \partial_{n_1}\tilde{u}_1 v_1 + \partial_{n_2}\tilde{u}_2 v_2 \mathrm{d}x_2 + b(-\lambda_0,v) + b(\xi,u-\tilde{u})}{\|\xi\|_{0,\Gamma} + |v|_1}, \tag{19}$$

which proves the bound from above in (16). Accounting for the relation between $\partial_{n_1}\tilde{u}_1$,$\partial_{n_1}\tilde{u}_2$ and λ_0 given in Lemma 2.1 we have:

$$\|e\| \le \frac{1}{\beta} \sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{-\int_\Gamma \xi(\tilde{u}_1 - \tilde{u}_2)\,dx_2}{\|\xi\|_{0,\Gamma} + |v|_1},$$

and finally

$$\|e\| \le \frac{1}{\beta}\|\tilde{u}_1 - \tilde{u}_2\|_{0,\Gamma} = \frac{1}{\beta}\|\tilde{u}_1 - \frac{1}{|\Gamma|}\int_\Gamma \tilde{u}_1 \,\mathrm{d}x_2\|_{0,\Gamma}. \tag{20}$$

Now, let us consider the second case where $f \in L^2(\Omega)$ and the mortar subspace is $\Lambda_2 = L^2(\Gamma)$. Starting from the inequality (19) accounting for results of Lemma 2.2 arguing in the same way as before, we get the following indicator:

$$\|e\| \le \frac{1}{\beta}\|\partial_{n_1}\hat{u}_1 + \partial_{n_2}\hat{u}_2\|_{0,\Gamma} = \frac{1}{\beta}\|\partial_{n_1}\hat{u}_1 - \frac{1}{|\Gamma|}\int_\Gamma \partial_{n_1}\hat{u}_1 \mathrm{d}x_2\|_{0,\Gamma}. \tag{21}$$

4. Optimization with respect to the location of Γ

Let a denote the position of the boundary Γ. Due to relation (20), the proposed strategy is to minimize the functional $J(a)$ with respect to a defined by

$$J(a) = \|\tilde{u}_1(a,x_2) - \frac{1}{|\Gamma_a|}\int_{\Gamma_a} \tilde{u}_1(a,x_2)\,\mathrm{d}x_2\|^2_{0,\Gamma}.$$

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The algorithm of minimization we propose is a simple descent algorithm. Let a_0 and Tol be fixed.

(1) evaluate the derivative $DJ(a_n)$
(2) if $|DJ(a_n)| \leq \text{Tol}$ stop and if not
(3) $a_{n+1} = a_n - \theta\ DJ(a_n)$ where θ is a fixed positive number.
(4) $n = n+1$ return to the beginning.

Now we evaluate numerically the derivative with respect to the location of the boundary Γ. To compute $DJ(a_n)$ define:

$$I(a, x_2) = \tilde{u}_1(a, x_2) - \int_0^1 \tilde{u}_1(a, x_2)\mathrm{d}x_2.$$

Observe that

$$J(a) = (I(a, x_2), I(a, x_2))_{L^2(\Gamma)},$$

its derivative $DJ(a) = 2\left(\frac{\partial I(a, x_2)}{\partial a}\right), I(a, x_2)_{L^2(\Gamma)}$ where:

$$\frac{\partial I}{\partial a}(a, x_2) = \frac{\partial \tilde{u}_1}{\partial a}(a, x_2) - \int_0^1 \frac{\partial \tilde{u}_1}{\partial a}(a, x_2)\mathrm{d}x_2.$$

To compute the derivative of $\tilde{u}_1$ with respect to $0 < a < 1$, the location of Γ_a we use the following change of geometry which consists in mapping the domain Ω with a moving boundary Γ_a onto a domain with a fixed boundary $\Gamma_{1/2}$. Thus the change of geometry will yield a change in coefficients of PDEs. Define the transformation T by

$$\begin{aligned} [0,1]\times[0,1] &\to [0,1]\times[0,1], \\ (z, x_2) &\to (x_1, x_2) = (T(z,a) = (2-4a)z^2 + (4a-1)z, x_2), \end{aligned}$$

thus the segment $\Gamma_{\frac{1}{2}}$ is mapped to Γ_a. The unknown ψ is defined by $\psi = U \circ T$. Equation (2) becomes:

$$\begin{cases} -D_zT.D^2_{zz}\psi - (D_zT)^3.D^2_{x_2x_2}\psi + D^2_{zz}T.D_z\psi = (D_zT)^3.f(T, x_2), \\ \partial_n\psi = 0 \text{ on } \gamma_{2i};\ 1 \leq i \leq 2, \\ \psi = 0 \text{ on } \gamma_{2i-1};\ 1 \leq i \leq 2. \end{cases} \tag{22}$$

A variational formulation for the decomposed domain problem corresponding to the problem (22) with a mortar subspace Λ_0 is: $\Omega_1 = (0, \frac{1}{2}) \times (0,1)$ and $\Omega_2 = (\frac{1}{2}, 1) \times (0,1)$;

$$\begin{cases} \displaystyle\sum_{i=1}^2 \int_{\Omega_i} c(z).\nabla\tilde{\psi}_i.\nabla v_i\,\mathrm{d}x_1\mathrm{d}x_2 + 2\sum_{i=1}^2 \int_{\Omega_i} D^2_{zz}T.D_z\tilde{\psi}_i.v_i\,\mathrm{d}x_1\mathrm{d}x_2 \\ \displaystyle\quad + \int_{\Gamma_a} \lambda(v_1 - v_2)\,\mathrm{d}x_2 = \sum_{i=1}^2 \int_{\Omega_i} (D_zT)^3.f_i(T, x_2).v_i\,\mathrm{d}x_1\mathrm{d}x_2, \quad \forall v \in V \\ \displaystyle\int_\Gamma \xi(\tilde{\psi}_1 - \tilde{\psi}_2)\,\mathrm{d}x_2 = 0, \quad \forall \xi \in \Lambda_0, \end{cases} \tag{23}$$

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where c is 2×2 diagonal matrix such as $c_{11}=D_zT$ and $c_{22}=(D_zT)^3$. Now we calculate the derivative of the indicator $J(a)$ with respect to a with function $\tilde{\psi}$:

$$DJ(a) = 2\int_0^1 \left(\tilde{u}_1(a,x_2) - \int_0^1 \tilde{u}_1(a,x_2)\,\mathrm{d}x_2\right)\frac{\partial I}{\partial a}(a,x_2)\,\mathrm{d}x_2,$$

therefore,

$$DJ(a) = 2\int_0^1 \left[\tilde{\Psi}_1 - \int_0^1 \tilde{\Psi}_1 \mathrm{d}x_2\right]\cdot\left[\tilde{\Psi}_{1a} - \int_0^1 \tilde{\Psi}_{1a}\mathrm{d}x_2\right]\mathrm{d}x_2,$$

where $\tilde{\Psi}_{ia}$ denotes the derivative with respect to a of function $\tilde{\Psi}_i$ for $1\le i\le 2$. In the case where $f\in L^2(\Omega)$ and $\Lambda_2 = L^2(\Gamma)$ we use the indicator given in (21), we have:

$$J(a) = \int_0^1 (\partial_{n_1}\hat{u}_1(a,x_2) - \int_0^1 \partial_{n_1}\hat{u}_1(a,x_2)\mathrm{d}x_2)^2\mathrm{d}x_2, \tag{24}$$

$$DJ(a) = 2\int_0^1 \left[\partial_{x_1}\hat{\Psi}_1 - \int_0^1 \partial_{x_1}\hat{\Psi}_1\mathrm{d}x_2)\right]\left[\partial_{x_1}\hat{\Psi}_{1a} - \int_0^1 \partial_{x_1}\hat{\Psi}_{1a}\mathrm{d}x_2)\right]\mathrm{d}x_2, \tag{25}$$

where $\hat{\Psi}_{ia}$ is the derivative of $\hat{\Psi}_i$ with respect to the variable a. Taking the derivative of the first equation of (22) with respect to the variable a we have:

$$\begin{aligned}&-D_zT\cdot D^3_{azz}\Psi - (D_zT)^3\cdot D^3_{ax_2x_2}\Psi + D^2_{zz}T\cdot D^2_{az}\Psi = D^2_{az}T\cdot D^2_{zz}\Psi + 3D^2_{az}\cdot(D_zT)^2\cdot D^2_{x_2x_2}\Psi\\ &\quad - D^3_{azz}T\cdot D_z\Psi + 3D^2_{az}T\cdot(D_zT)^2\cdot f(T,x_2) + D_aT(D_zT)^3\cdot D_{x_1}f(T,x_2).\end{aligned} \tag{26}$$

Defining $\Psi_a = D_a\Psi$ Equation (26) becomes:

$$\begin{cases} -D_zT\cdot D_{zz}\Psi_a - (D_zT)^3\cdot D^2_{x_2x_2}\Psi_a + D^2_{zz}T\cdot D_z\Psi_a = D^2_{az}T\cdot D^2_{zz}\Psi + 3D^2_{az}T\cdot(D_zT)^2\cdot D^2_{x_2x_2}\Psi\\ -D^3_{azz}T\cdot D_z\Psi + 3D^2_{az}T\cdot(D_zT)^2\cdot f(T,x_2) + D_aT(D_zT)^3\cdot D_{x_1}f(T,x_2),\\ \partial_n\Psi_a = 0, \text{ on } \gamma_{2i};\ 1\le i\le 2,\\ \Psi_a = 0, \text{ on } \gamma_{2i-1};\ 1\le i\le 2. \end{cases} \tag{27}$$

A variational formulation for problem (27) in decomposed domain setting is:

$$\begin{cases} \displaystyle\sum_{i=1}^2 \int_{\Omega_i} (c(z)\nabla\tilde{\psi}_{ia}\nabla v_i + 2D^2_{zz}TD_z\tilde{\psi}_{ia}v_i)\mathrm{d}x_1\mathrm{d}x_2 + \int_{\Gamma_a}\lambda(v_1 - v_2)\,\mathrm{d}x_2\\ \quad \displaystyle = \sum_{i=1}^2 \int_{\Omega_i}(-c_a(z)\nabla\tilde{\psi}_i\nabla v_i - 2D^3_{azz}TD_z\tilde{\psi}_iv_i)\mathrm{d}x_1\mathrm{d}x_2\\ \quad\quad \displaystyle + \sum_{i=1}^2 \int_{\Omega_i}[3D^2_{az}T(D_zT)^2 + D_aT(D_zT)^3D_{x_1}]f_i(T,x_2).v_i\mathrm{d}x_1\mathrm{d}x_2, \quad \forall v\in V,\\ \displaystyle\int_\Gamma \xi(\tilde{\psi}_{1a} - \tilde{\psi}_{2a})\,\mathrm{d}x_2 = 0, \quad \forall\xi\in\Lambda_0, \end{cases} \tag{28}$$

where c_a is a 2×2 diagonal matrix such as $c_{a11} = -D^2_{az}T$ and $c_{a22} = -3D^2_{az}T(D_zT)^2$.

Now this section is ended with some numerical examples. Let U be defined by

$$U(x_1,x_2)=\begin{cases} x_1\left[\left(x_1-\frac{1}{2}\right)^3\cdot(x_2-x_2^2)^2+(1-x_1)^2\right], & 0\le x_1\le\frac{1}{2},\\ x_1(1-x_1)^2, \quad \frac{1}{2}\le x_1\le 1, & \end{cases} \tag{29}$$

and function f by

$$f(x_1,x_2)=\begin{cases} (12t-2)x_1^4+(-18t+3)x_1^3+\left(\frac{-3}{2}+9t-12t^2\right)x_1^2+\left(9t^2-\frac{3}{2}t-\frac{23}{4}\right)x_1\\ -\frac{3}{2}t^2+4, \quad 0<x_1<\frac{1}{2},\\ -6x_1+4, \quad \frac{1}{2}<x_1<1, \end{cases} \tag{30}$$

with $t=(x_2-x_2^2)^2$. It is straightforward to check that U solves

$$\begin{cases} -\Delta U(x_1,x_2)=f(x_1,x_2), \text{ in } \Omega,\\ \partial_n U=0, \text{ on } \gamma_{2i},\ 1\le i\le 2,\\ U=0, \text{ on } \gamma_{2i-1},\ 1\le i\le 2. \end{cases} \tag{31}$$

Observe that U solves a domain decomposition formulation with an interface Γ located at $a=\frac{1}{2}$.

Let us define V_h (respectively, W_h) as the space that approximates V (respectively, W) with a triangular Lagrange finite element method of order one. The maximum size of triangle's diameter is $h=10^{-1}$. The space Λ_{2_h} consists of the traces of space V_h on the interface Γ_a. Let $(u_h,\lambda_h))\in V_h\times\Lambda_{2_h}$ be the solution to

$$\begin{cases} \sum_{i=1}^{2}\int_{\Omega_i}\nabla u_{i_h}\nabla v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2+\int_\Gamma \lambda_h(v_{1_h}-v_{2_h})\,\mathrm{d}x_2=\sum_{i=1}^{2}\int_{\Omega_i} f v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2, \quad \forall v_h\in V_h,\\ \int_\Gamma \xi_h(u_{1_h}-u_{2_h})\,\mathrm{d}x_2=0, \quad \forall \xi_h\in\Lambda_{2_h}. \end{cases} \tag{32}$$

Figure 1 shows the error between U and u_h.

The discontinuity of the error through Γ is due to the discontinuity of the approximate solution through Γ.

Now let us define $(\tilde{u}_h,\lambda_0))\in V_h\times\{C^{\mathrm{st}}\}$ be solution to

$$\begin{cases} \sum_{i=1}^{2}\int_{\Omega_i}\nabla \tilde{u}_{i_h}\nabla v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2+\int_\Gamma \lambda_0(v_{1_h}-v_{2_h})\,\mathrm{d}x_2=\sum_{i=1}^{2}\int_{\Omega_i} f v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2, \quad \forall v_h\in V_h\\ \int_\Gamma \xi(\tilde{u}_{1_h}-\tilde{u}_{2_h})\,\mathrm{d}x_2=0, \quad \forall \xi\in\{C^{\mathrm{st}}\}. \end{cases} \tag{33}$$

Figure 2 shows the error between the exact solution and the solution to the domain decomposition problem (33) for four locations of Γ_a.

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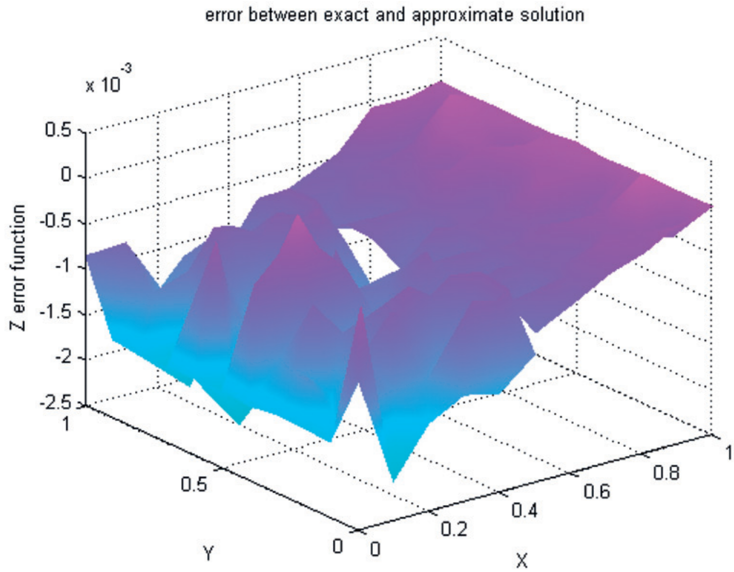

Figure 1. Error function.

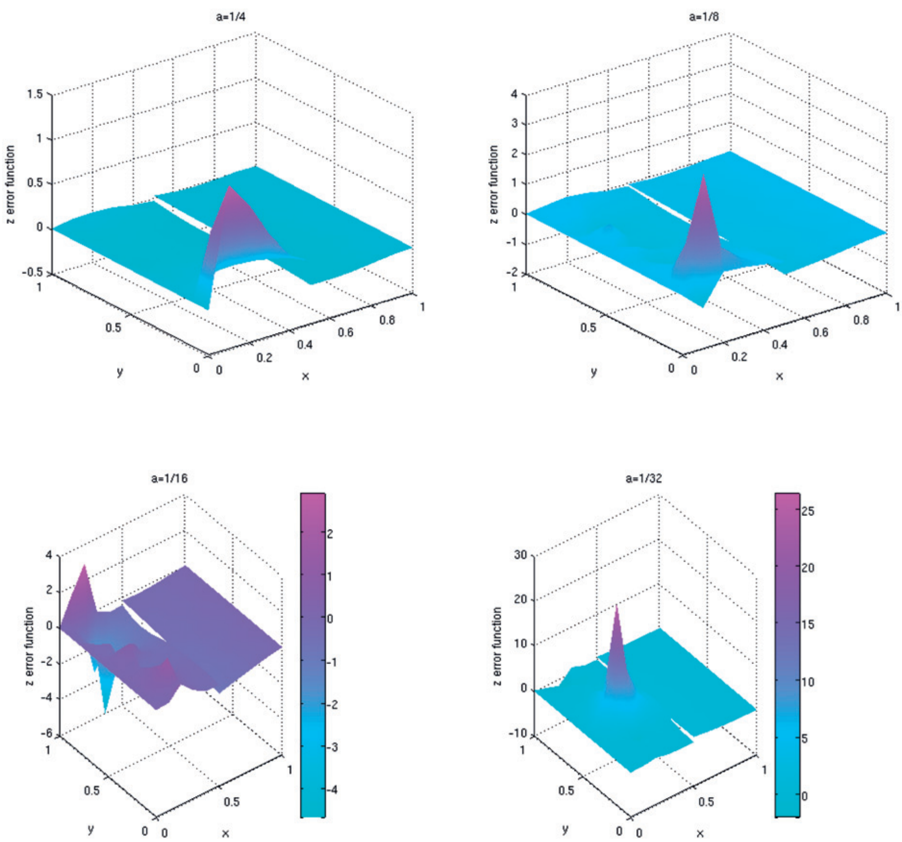

Figure 2. Error distribution for four locations of Γ_a.

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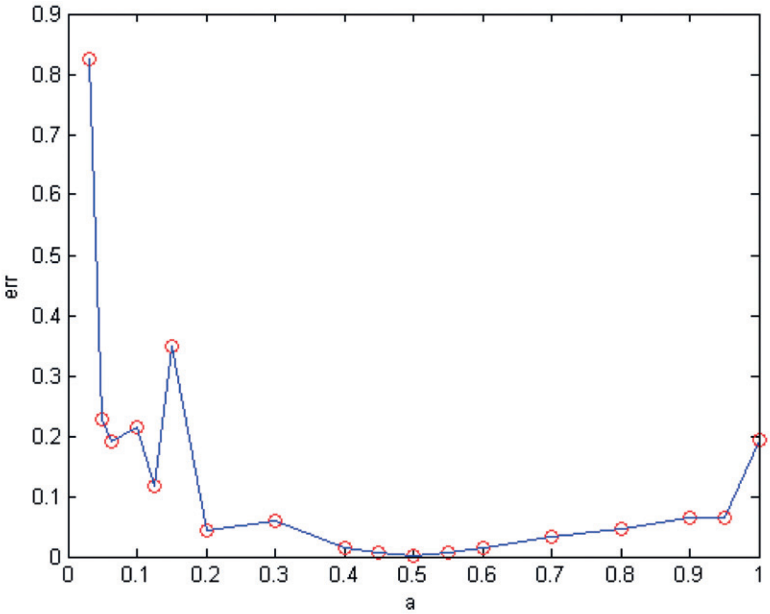

Figure 3. Indicator for mortar subspace Λ_0.

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Define the indicator by

$$J(a) = \|\tilde{u}_{1_h}(a, x_2) - \frac{1}{|\Gamma_a|}\int_{\Gamma_a} \tilde{u}_{1_h}(a, x_2)\,\mathrm{d}x_2\|^2_{0,\Gamma}.$$

Problem (28) is approximated in $V_h \times \{C^{\mathrm{st}}\}$, then $J(a)$ is computable whatever a is. In Figure 3, the curve $a \mapsto J(a)$ is presented.

Algorithm (1)–(4) has been implemented, and the derivative $DJ(a_n)$ has been computed by solving problem (28) approximated with a triangular Lagrange finite element method of order one. Figures 4 and 5 show convergence curves for starting points $a_0 = 0.35$ and $a_0 = 0.65$ with a mortar subspace Λ_0.

Let $f \in L^2(\Omega)$ and $((\hat{u}_h, \lambda_2) \in W_h \times \Lambda_{2_h}$ be solution to

$$\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{2}\int_{\Omega_i} \nabla\hat{u}_{i_h}\nabla v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2 + \int_\Gamma \lambda_2(v_{1_h} - v_{2_h})\,\mathrm{d}x_2 = \sum_{i=1}^{2}\int_{\Omega_i} f v_{i_h}\,\mathrm{d}x_1\mathrm{d}x_2, \quad \forall v_h \in W_h, \\ \displaystyle\int_\Gamma \xi(\hat{u}_{1_h} - \hat{u}_{2_h})\,\mathrm{d}x_2 = 0, \quad \forall \xi \in \Lambda_{2_h}. \end{array}\right. \tag{34}$$

Figure 6 shows the error between the exact solution and the approached solution.

Define the indicator by

$$J(a) = \int_0^1 \left(\partial_{n_1}\hat{u}_{1_h}(a, x_2) - \int_0^1 \partial_{n_1}\hat{u}_{1_h}(a, x_2)\mathrm{d}x_2\right)^2 \mathrm{d}x_2. \tag{35}$$

Figure 7 shows this indicator plotted as function of the location of the interface.

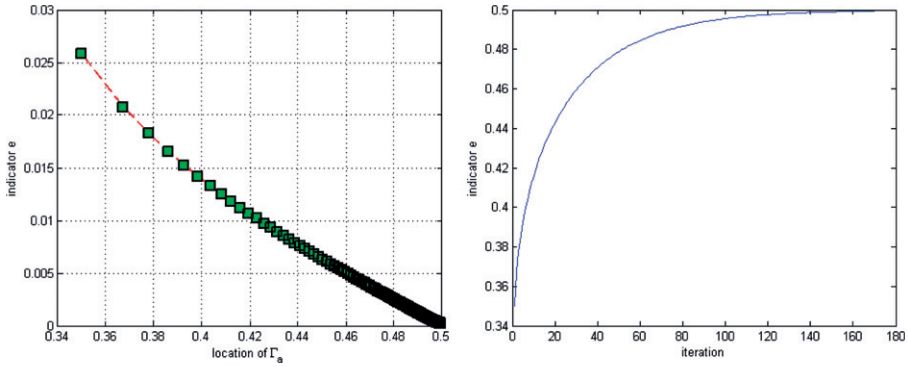

Figure 4. Indicator as function of location of the interface (left) and position of the interface as function of iterations (right).

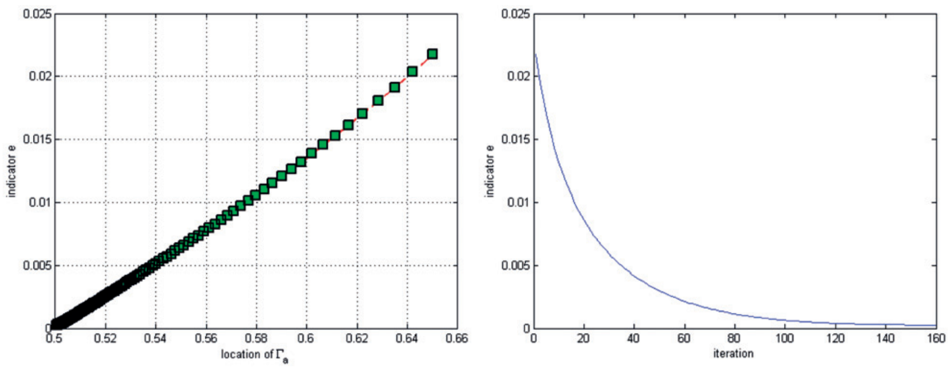

Figure 5. Indicator as function of location of the interface (left) and location as function of iterations (right).

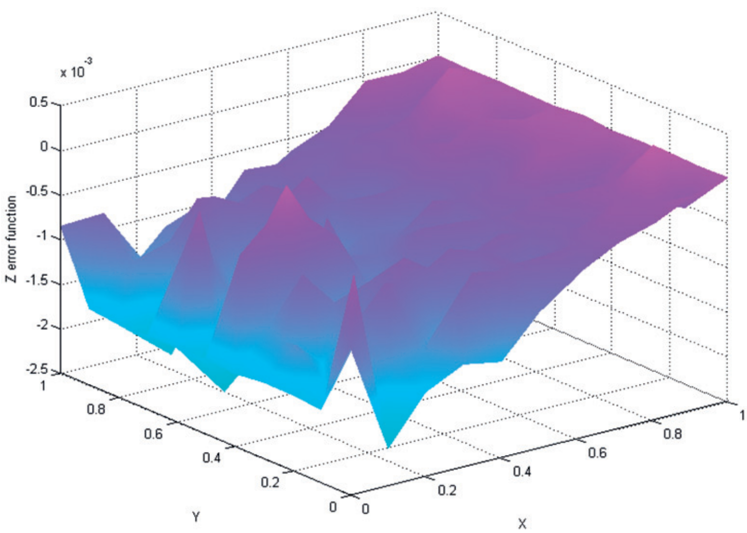

Figure 6. Error with a mortar subspace Λ_2.

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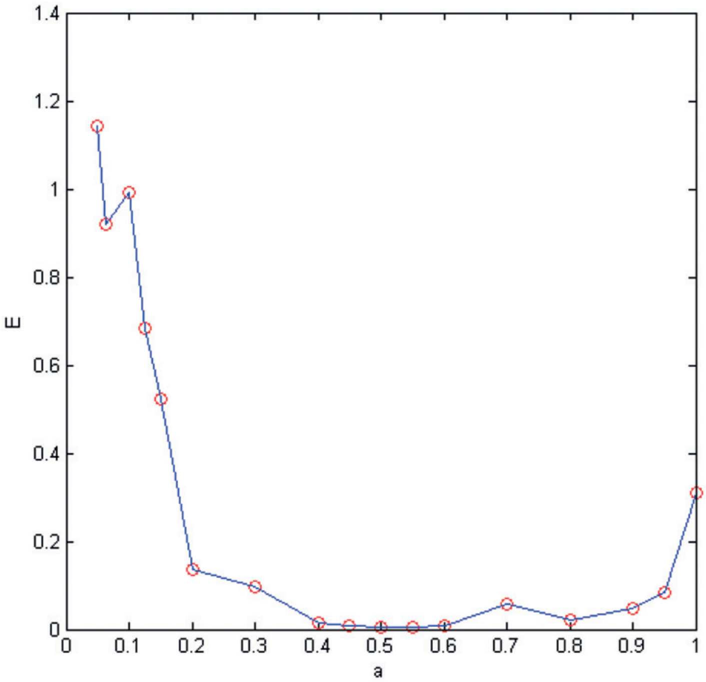

Figure 7. Error evaluation in case Λ_2.

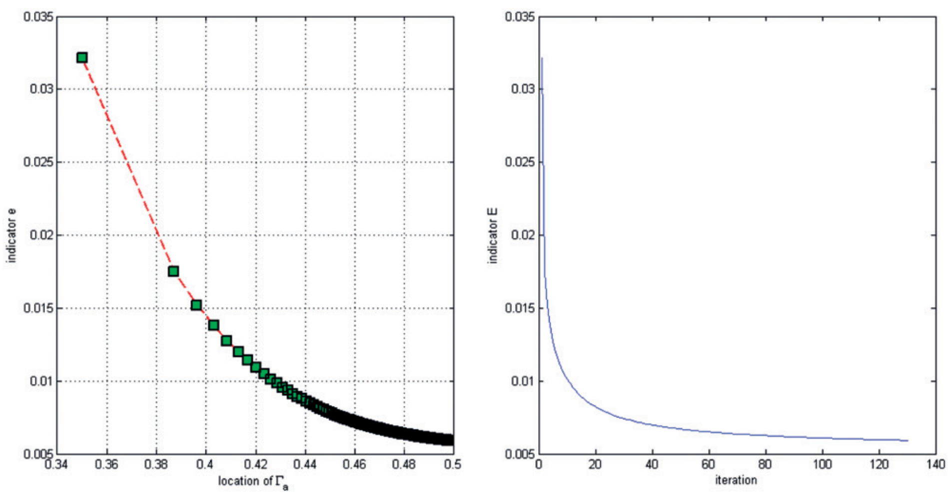

Figure 8. Indicator as function of location (left) and position of the interface as function of iterations (right).

Proceeding in the same way as before, Figure 8 shows the indicator represented as function of the position of the interface for a starting position $a_0 = 0.35$, and the location of the interface is described as function of the iterations.

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Table 1. Location indicator and mesh error indicator for two meshes.

h	a	Location indicator	Minimum error indicator mesh	Maximum error indicator mesh
10^{-1}	0.45	0.0064	0.0064	0.0308
	0.40	0.0139	0.0051	0.0243
	0.35	0.0258	0.0021	0.0254
5×10^{-2}	0.45	0.0064	0.0013	0.0082
	0.40	0.0140	0.0012	0.0065
	0.35	0.0265	0.00041539	0.0069

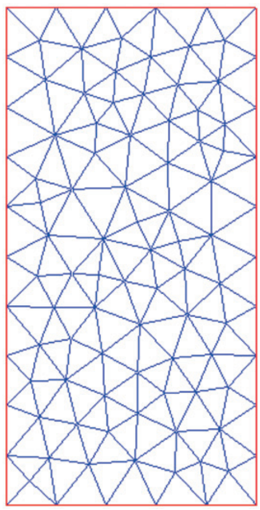

Figure 9. Mesh of domain Ω_1.

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Let us conclude this article with some computational considerations.

- The proposed indicator is computed only with u_{h_1}, the approximate solution in domain Ω_1.
- When dealing with a 2D or 3D domain Ω_1 linked to 1D domains, that is to say when considering a PDE linked with ODEs, one can increase the accuracy by either expanding the PDE domain (inherently more accurate than the 1D domains, in general) or by refining the mesh in Ω_1. A crucial question is which of these alternatives is most cost effective. The answer is quite simple. By using your favourite indicator of the mesh error, compare it with the indicators of the location error proposed in this article. Then you are able to decide whether you should refine the mesh or you should move the interfaces. Assume for example that for problem 33 an accuracy of 10^{-2} is required. Let us start with an interface located at $a = 0.35$ and with a size of mesh of 10^{-1}. Computing the indicator of the location error we

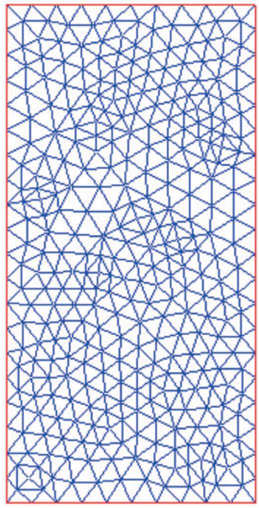

Figure 10. Refined mesh of domain Ω_1.

get 0.0258, and the indicator of the mesh error is valued between 0.0034 and 0.024 (Table 1). Thus the interface is moved to $a = 0.4$ in order to enlarge the size of the domain Ω_1. The indicator of the location error becomes 0.0139, and the indicator of the mesh error is valued between 0.0051 and 0.0243. The mesh is then refined in the domain Ω_1 with a mesh size of 5×10^{-2} and the indicator of the mesh error is valued between 0.0012 and 0.0065. The interface is now moved to $a = 0.45$, the indicator of the location error becomes 0.0064. Figure 9 shows the mesh of domain Ω_1. The mesh refinement strategy is quite crude (Figure 10), since the mesh is uniformly refined.

- Observe that whatever the values of the indicator of the mesh error is, it is possible to reach the optimal location of the interface.

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